Choice, ranking and circularity in asymmetric relations

Monsuur, Herman

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IN

ASYMMETRIC RELATIONS

HERMAN MONSUUR

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ASYMMETRIC RELATIONS

Proefschrift

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Promotor:

Prof. dr. S.H. Tijs

Co-promotores:

Prof. dr. ir. R. Delver

Dr. A.J.A. Storcken

A just balance and scales are the LORD's,
all the weights in the bag are His work.

Proverbs 16: 11.

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CONTENTS

PREFACE

In this monograph we study pairwise comparison structures. In these pairwise comparisons, alternatives are evaluated with regard to certain objectives. These evaluations may be in terms of e.g. preference, importance, dominance, quality or effectiveness. We assume that whenever in the comparison between two alternatives a strict choice for one of them is uncovered, this choice is definite. This occurs for example when alternatives and criteria have become final, and choices can easily be motivated. But also in situations where we have to meet a deadline: in voting procedures, such as majority voting with a tie-breaking rule, or when we stand at the eve of an important decision.

These strict choices result in asymmetric relations on the alternatives, which frequently occur in management and social sciences. For instance, these asymmetric relations, consisting of the strict choices, may represent hierarchies, preferences or dominance structures, compare Berge and Duchet (1991), Miller (1980), Rapoport (1983), Roberts (1979) and van Deemen (1991).

Using the pairwise comparisons, hence the asymmetric relation, we want to arrange the alternatives in a certain order. If the asymmetric relation reflects a natural order of the alternatives, this is easy. But, because of intransitivities in the pairwise comparisons, this arranging can be very complicated. It is possible that there is no winner, no best option, notwithstanding our ability to make a strict choice in the comparisons between each couple of alternatives. These intransitive choices may be due to circular reasoning, majority voting, the selection of the criteria used in each comparison, or other, more hidden factors.

Yet, almost always there is need for an ordering of the possible alternatives. We want to arrange players from best to worst, or determine a winning coalition in political situations. Therefore, in this thesis, our first main objective is: how to rank alternatives, how to choose a best alternative from a set of pairwise ordered alternatives, how to determine a dominating coalition? To solve this problem, various ranking rules and choice functions are proposed and discussed in literature. Among them we find obvious methods based on scores, but also methods that are based on game-theoretical or strategic considerations. In general, it is hard to decide which method has to be used. Each of the procedures considered has some appealing features.

Of course, one may take their domain of applicability into consideration. A method that is designed to deal with one category of ranking situations, is not automatically suited for ranking asymmetric relations resulting from another category. For example, a rule that ranks players in a round-robin tournament, may perform badly when applied to problems arising in a more political context, where the notion of a coalition becomes important.

This inspired our approach, which is the axiomatic approach. To decide between various procedures, we start a qualitative study by developing and investigating various conditions or criteria. These conditions are described in mathematical terms and, depending on the ranking situation, represent to a more or less extent the notions of reasonableness and fairness. As such, they are applicable to for instance management science, political science and measurement theory. Most conditions are so-called independence conditions, such as the well-known Arrow's independence of irrelevant alternatives condition. We may roughly divide them in two groups: conditions relating variable sets of alternatives to the final ranking or choice, and on the other hand conditions concerning variable asymmetric relations, but with fixed alternative set. We verify these conditions for ranking rules and choice functions. In some cases we are able to derive characterizations. A characterization is a collection of independent conditions for a class of methods, such that there remains just one, precisely described method satisfying these conditions. In this way, we may recommend and motivate use of a method in a particular situation, hence determining their domain of applicability. Related research is described in e.g. Bouyssou (1992), Henriet (1985), Laffond, Laslier and Le Breton (1993), Moulin (1986), and many others.

The second problem that is addressed to in this thesis, concerns the inconsistency or circularity of tournaments. Decision theory is partly aimed at structuring and clarifying subjective reasoning in order to make better decisions. Therefore, we would like to quantify the quality of someone's judgment. Or we want to determine the usability of a set of pairwise comparisons. This is particularly important because if consistency is low, the rankings provided by the various ranking procedures will vary more. The inconsistency measures that are in use, point out to what extent a set of preferences can be sorted out, or determine the deviation to an ordering. Just as is the case for ranking rules and choice functions, there does not exist an inconsistency measure which is recognized or generally accepted as being best. In the scientific literature one concentrates on the statistical aspects of these measures, see e.g. Bezembinder (1980), David (1988) or Slater (1961). Our approach is axiomatic. We characterize circularity measures by necessary and sufficient conditions.

OUTLINE OF THE MONOGRAPH

Chapter 1 splits into three parts. The first section discusses examples illustrating the issues of ranking, choice and circularity. We give several ranking and choice methods and also discuss some conditions that we use to evaluate the performance of a method. Next to it, we discuss two levels of analysis, a local and a global level. It will appear that many methods and conditions are focused on one of these two levels of analysis. The second section contains definitions and some preliminaries. We introduce basic mathematical notions that are used throughout: binary relations, tournaments, linear and weak orders, transitive closure, etc. The chapter is concluded with some preliminary results concerning irreducibility and cycles. An index of methods and criteria used in the comparison between them, is presented in an appendix to this chapter.

Chapter 2 is devoted to a study of choice functions on tournaments (complete asymmetric relations). A choice function assigns to any tournament on a set of alternatives X a nonempty subset of X, mostly called the choice set or set of winners of the tournament. Starting with some elementary properties for these functions, we discuss and characterize the top cycle of a tournament. Since this cycle often is unmanageably large and may contain Pareto dominated alternatives if the tournament stems from pairwise majority voting, other choice functions are considered. For example, we discuss the uncovered and the minimal covering set and characterizations for them. We also consider the Banks set, the tournament equilibrium set, the Copeland choice set and the Slater choice set. An alternative proof for the unicity of the optimal strategy of the tournament zero-sum game is given and, as an application of this proof, we present a characterization of the choice function corresponding to this game. After that, we introduce, discuss and verify a few new conditions, such as Δ-IIA, separability, global monotonicity and tail stability. We conclude the chapter with a table of set-theoretical inclusions and a table of properties for a few choice functions.

In chapter 3, we study ranking rules. These are rules that map a tournament onto a weak order, hence rank the alternatives from best to worst. We introduce lots of new conditions for the evaluation of the performance of a ranking rule. They are used in characterizations. For example, we give a characterization of the set of ranking rules that are induced by a choice function. We derive two different characterizations of the ranking rule that assigns to a tournament its transitive closure, and a characterization of the Copeland ranking rule. We also give a characterization of the Copeland choice function and a related one. Among the rules that we compare are those induced by a notion of distance between relations and those induced by ratings. Aside from new conditions used in the characterizations, we introduce and consider two conditions illustrating the difference between two levels of analysis. The results of the systematic comparison of ranking rules is summarized in a table at the end of the chapter. In the first appendix, we consider for a few ranking rules maximal subsets of the set of tournaments, on which these rules satisfy the Δ-IIA condition. In the second appendix, we consider the relation between the inconsistency index developed by Saaty in his 'Analytic Hierarchy Process' and the insertion of new alternatives.

In chapter 4, we discuss solutions for dominance structures. A well-known solution scheme is formed by the von Neumann-Morgenstern stable sets (vNM-stable sets). It combines the notions of internal stability and external domination for a coalition. The first main result is a characterization of the choice function corresponding to the vNM-stable set, in case of acyclic relations which for instance occur in hierarchies. In some cyclic relations there is no vNM-stable set. This led us to introduce a generalization. Using game-theoretical arguments, we obtain our second main result of this chapter; a characterization of a generalized vNM-stable set satisfying a particular balanced domination property. Besides that, we give some set-theoretical comparisons between four choice functions for dominance structures and verify some independence conditions that are significant in this context.

Chapter 5 handles our second problem, the circularity of a tournament. The emphasis lies on a comparison of two well-known measures: the number of 3-cycles λ and Slater's i. As already remarked, our approach is axiomatic. We show, using an impossibility theorem, that there is no circularity measure that combines all attractive features of both λ and i. Therefore, we present characterizations for each of them and consider the problem of when to use which measure. Furthermore, we characterize circularities that are defined using scores of the alternatives and show its connection to the (probabilistic) strict utility model. Next to it, we characterize a circularity measure that is induced by the transitive closure of a tournament. Following a statistical analysis, a systematic study of properties and measures is summarized in a table at the end of the chapter.

NOTATIONS AND SYMBOLS

$\mathbb{N} = \{1,2,3,...\}$: set of positive integers
$\mathbb{Z} = \{...,-1,0,1,2,...\}$: set of integers
$\mathbb{R}$: set of real numbers
$\mathbb{R}^t$: set of t-tuples of real numbers, $t \in \mathbb{N}$

$|x|$: absolute value of x
$|V|$: number of elements of V, cardinality of V
$\binom{n}{k}$: binomial coefficient $n!/\{k!(n-k)!\}$
$\wedge$: and
$\neg$: not
Σ : sum
Π : product
$\circ$: composition
$\emptyset$: empty set

2^X : set of subsets of X
$X \times X$ or X^2 : Cartesian product of the set X with itself
$X \times Y$: Cartesian product of the sets X and Y
$X \cup Y$: union of X and Y
$X \cap Y$: intersection of X and Y
$X \subseteq Y$: X is contained in Y
$X \setminus Y$: elements of X that are not in Y
X_x : $X \setminus \{x\}$

$\mathbb{T}$, $\mathbb{T}(X)$, $\mathbb{T}^*$: set of tournaments 13, 25, 82
$\mathbb{G}$, $\mathbb{G}(X)$: set of asymmetric relations 13, 25
$\mathbb{A}$, $\mathbb{A}(X)$: set of acyclic asymmetric relations 117
$\mathbb{L}$, $\mathbb{L}(X)$: set of linear orders 14
$\mathbb{W}$, $\mathbb{W}(X)$: set of weak orders 14

T or (T,X) : tournament on X
G or (G,X) : asymmetric relation on X
R or (R,X) : relation on X
$S(G)$ or $S(G,X)$: choice set of (G,X)
$f(G)$ or $f(G,X)$: ranking of (G,X)
$\gamma(T)$ or $\gamma(T,X)$: circularity of (T,X)

$R|Y$ or (R,Y) : restriction of R to Y 12
τR or $\tau(R,X)$: transitive closure of R 14
aR or a(R,X) : asymmetric part of R 15
vR or v(R,X) : converse of R 73
σR or $\sigma(R,X)$: permutation of R 15
Max(R) or Max(R,X) : maximal elements of R 16
Best(R) or Best(R,X) : best elements of R 16
N(T) or N(T,X) : nearest adjoining orders to (T,X) 39

$(a,b) \in$ T, R or G : a is as least as good as b in T, R or G 12
aTb, aRb or aGb : a is as least as good as b 12

» (T,A) » (T,B) : concatenation of (T,A) and (T,B) 16
if W is a weak order then
a » b : W means: a is ranked ahead of b 14
A » B : W means: for all $a \in A$, $b \in B$: a » b : W 14

$\geq$ $x \geq y$: W, W a weak order: x is as least as good as y 14
$\cong$ $x \cong y$: R : x and y are indifferent in R 12

yxT : $T \setminus \{(x,y)\} \cup \{(y,x)\}$ 158
rT or <abc>T : reverse orientation of 3-cycle r in T 170
$s_x(T)$ or out(x,T) : Copeland score of alternative x 19, 103
in(x,T) : number of alternative that are better than x 104
upp(x,T) : set of elements that are better than x in T 78
low(x,T) : set of elements that are worse than x in T 78

γ_{scaled} : scaled circularity measure 183

$d(p,q)$: distance between relations p and q 156

$r(\alpha,\beta)$: rank correlation between relations α and β 189

λ : number of 3-cycles 154

i : Slater's i 156

χ : number of 4-cycles 173

δ : proportion of preferences on a cycle 173

ρ : Bezembinder's ρ 173

sc : $d(f_{out}(T),T)$ 173

rsc : $d(f^{*}_{out}(T),T)$ 189

ε : number of cyclic preferences 179

supp($\underline{p}$) : support of $\underline{p}$

$\underline{p}^{t}$: transpose of $\underline{p}$

K^{n} : set of n-strategies 41

O(G) : set of optimal strategies for the G-game 42,45

rel(O(G)) : = $\{\underline{p} | Y: \underline{p} \in O(G)\}$, where Y is the smallest set such that for all $\underline{q} \in O(G)$: supp($\underline{q}$) $\subseteq$ Y 126

int(rel(O(G))) : interior of rel(O(G)) 126

CHAPTER 1

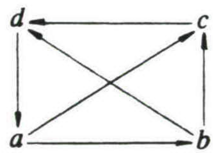
INTRODUCTION

This chapter is divided into three parts. First of all, we illustrate *the issue* of ranking, choice and circularity. The second section gives *definitions* and some *preliminary* results concerning the relation between irreducible tournaments and the transitive closure. The appendix to this chapter contains an *index* of choice functions, ranking rules and circularity measures that are discussed in this thesis. Moreover, we list the conditions used in the comparisons between them.

1.1 THE ISSUE

Consider the following tournament T_1, which may be the result of 6 matches between four tennis players, *a*,*b*,*c* and *d*. For example, player *a* has beaten player *b* and *c*, but had to acknowledge player *d* as his superior.

T_1



Both player *a* and *b* have the highest score. Since player *a* has beaten player *b*, we may declare *a* to be winner of this small tournament. Moreover, we may rank the players to {a} » {b} » {c} » {d}, where the symbol '»' stands for: 'is ranked ahead of'.

In this tennis tournament, each couple of players is playing just one match. In such ranking situations we may distinguish two levels of analysis. A binary level where alternatives are compared in pairs, corresponding to the direct confrontations, and a more global level. We shortly discuss this global level. Since the tournament consists of once-only confrontations, it has a clear end in terms of time. Afterwards, when the matches are over, the tournament is experienced as a synthesis of the pairwise comparisons at the binary level, and is ranked accordingly, as is the case in most sports. It is generally accepted that in such

tournaments, it may happen that two alternatives are distinguishable at the binary level and yet are equivalent at the global level. More than that, for two alternatives *a* and *d*, it is possible that in the direct comparison the alternative *d* beats *a* while, because of insight obtained from global considerations, *a* is ranked higher than *d*. This may be the case when for example *a* has beaten more players than *d*, as in T_1. Since everyone agrees with the fact that tomorrow things may be quite different, no player is really bothered by these phenomenons.
The rule we used to determine a winner of this tennis tournament is very suited for this category of tournaments, because it stimulates the fighting spirit. In chapter 3, we introduce rules that reward player *d* for having beaten the strong player *a*.

To further illustrate the local and global level of analysis, consider the following example.
Suppose you own a restaurant. At a certain moment there are 3 guests. They all want to have the same dinner. You can offer them the choice between dinners a, b and c. The three guests reveal the following preference:

guest 1: a » b » c
guest 2: b » c » a
guest 3: c » a » b.

Assuming that the choice is up to you, which dinner has to be served? If you look at the majority in each pairwise comparison of the available dinners, you will reach the decision: a » b (because of guests 1 and 3 versus guest 2), b » c and c » a, a cyclic pattern or *intransitivity*, see figure 1.1. To Fishburn (1970), it illustrates the untenability of the transitivity condition, which excludes these cyclic patterns, as a general desideratum for social choice functions.

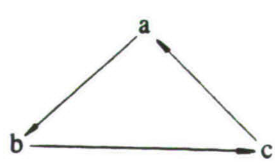


Figure 1.1 A circular tournament.

Note that this example clearly illustrates the need for a global level of analysis. Indeed, from a more global viewpoint, we see that the alternatives 'play the same role'. This may be seen with the help of figure 1.1: there does not exist a reasonable scale of measurement that can treat the three alternatives as being really different. Hence, instead of circulating forever between the three possible choices, each time striving for a better alternative, we switch to a higher level of analysis. We conclude that the three alternatives are equivalent.
Let us turn back to the tournament T_1 and suppose that you are a diplomat of the United Nations. Your job is to find a new security concept for a region that is teared into pieces because of internal conflicts. After long talks with the different parties, four feasible concepts emerge, labelled *a*, *b*, *c* and *d*.
Concept *x* is supposed to beat concept *y* in the pairwise comparison, if the majority of parties involved prefer concept *x*. Suppose that the votes give the tournament T_1 depicted above.
Which concept is the best?

If you propose concept *a* as solution of the conflict because it beats two other concepts, hence using the same rule that ranked the tennis players, you risk your life because concept *d* clearly is a better alternative for the majority of the parties involved. Similarly, any other single proposal is rejected. What causes this rejection?

In any case, the outcomes of this kind of tournaments are not the result of a once-only confrontation. In a manner of speaking, the tournament is continuously played, the comparisons between all proposals *x* and *y* are raised permanently and cannot enter on an agenda. Because of these permanent comparisons at the binary level, it is difficult to let the global analysis prevail the choices made there. This means that it is difficult to choose a winner, because at the binary tournament level, there always is a better choice due to the four binary comparisons: *a* better than *b* to be written as aT_1b, bT_1c, cT_1d and dT_1a, that constitute a cycle.
Instead of choosing just one alternative, one has to look for diplomatic compromises. Of course, in a compromise one may use all alternatives of the **top cycle**. This set is made up of alternatives that beat directly or

indirectly every other alternative. In our case, the top cycle is equal to $\{a,b,c,d\}$. But in our view, proposal c is not very attractive. Only when compared with d, this option has some advantages. But proposal d is also worse than b, which moreover majorizes proposal c. To use terminology introduced in chapter 2, proposal c is not uncovered; it is covered by b. An alternative x is uncovered, if there is no alternative y that is superior to x and beats all alternatives beaten by x. In the example above, the **uncovered set** is $\{a,b,d\}$. Solutions of this kind of choice problems are discussed in chapter 2 and 4.

So, we may distinguish several ranking and choice situations, each requiring a unique approach and treatment. Following three informal definitions, we present a few other examples, illustrating the issues of ranking, choice and circularity.

Informal definitions

A choice function S assigns to each asymmetric relation or tournament G a so-called choice set S(G), to be interpreted as: winners or best elements.

In this thesis, we also consider several ranking rules. A ranking rule f ranks the alternatives of X in a binary relation on X, denoted by (G,X), from best to worst. In general, we represent the outcome of a ranking rule f by $f(G,X) = X_1 \gg X_2 \gg \dots \gg X_n$, where for all $k \in \{1,\dots,n\}$, the alternatives from the sets X_k are on a level with each other, so-called maximal indifference classes, and the elements of X_1 are ranked above the elements of X_2 and so on, as was done with the tennis players above.

In chapter 5 of this monograph, we study circularity measures. These measures indicate the circularity or inconsistency of a tournament.

Example 1.1 *A betting problem*

Consider the following example, discussed in Williams (1982). Suppose you are born in the time of King Arthur and his knights of the round table. It was the time that brave man were used to challenge each other to a fight on lance and horses. At a certain moment there are 7 knights in town. The present ranking is as follows. Although lately most tilting have been indecisive, you may rely on Tristam to beat Lamarok and Gareth, Tor to beat

Tristam and Kay, etc, as is presented in figure 1.2. Which knight do you want to place money on?

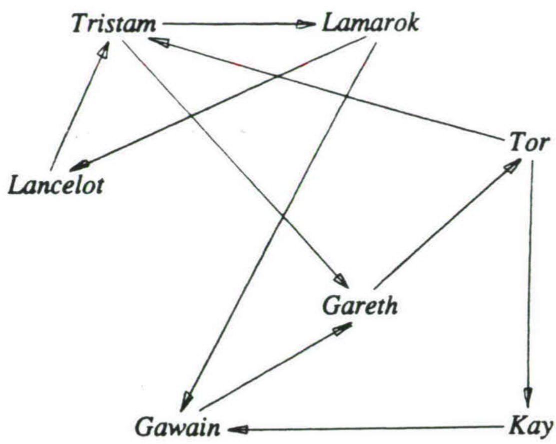


Figure 1.2 The present ranking of the knights.

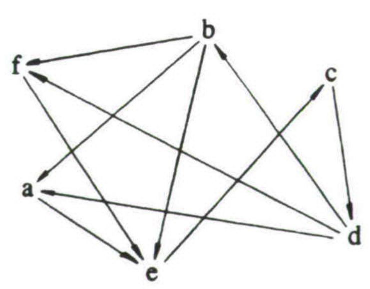
To introduce competition, we assume that there is another person. Each day, both of you may place a bet on one knight. For example, if you choose Gareth, while the other chooses Tor, you will receive 1 point, your opponent looses 1 point. What strategy has to be followed?
A solution of this problem is given by the **strategy equilibrium**, discussed in chapter 2 and 4. If you want at least to get even with your opponent, you may use the following strategy: never bet on Kay and Gawain, choose with equal probability one of the other knights.

Example 1.2 *A ranking problem*
A simple procedure, long used in chess tournaments (see David 1988, page 105), is to replace the score, which is the number of matches won, by the sum of the scores of players defeated.
Suppose that we have 5 players, a,b,c,d and e. The outcomes are as in the matrix below, where a '1' at matrix position (a,b) means that a has beaten b, and '0' means that b has beaten a.

	a	b	c	d	e
a	-	1	1	1	0
b	0	-	1	1	0
c	0	0	-	1	1
d	0	0	0	-	1
e	1	1	0	0	-

The scores of the players are equal to 3, 2, 2, 1, 2 respectively. If we rank the players to the sum of the scores of the players they defeat, we obtain the ranking {a,e} with 5 points, followed by {b,c} with 3 points and {d} with 2 points. A generalization of this procedure, the ranking rule $\mathbf{f}_{\mathbf{eigen}}$, due to Kendall and Wei, is discussed in chapter 3. It comes down to choosing a set of initial scores or ratings for the alternatives, such that the process described above, which consists of assigning to each player the sum of scores of the opponents defeated, does not change the relative ratings. Consider the initial rating 0.267, 0.174, 0.196, 0.127, 0.236. One may verify that, if we assign to each player the sum of ratings of the opponents defeated, we obtain the rating 0.497, 0.323, 0.363, 0.236, 0.441. But, if we scale these values, such that they sum up to 1, we regain our initial ratings.
The corresponding ranking is {a} » {e} » {c} » {b} » {d}.

Example 1.3 *A dominance problem*
In politics, it is almost always impossible to win by staying alone. Coalitions must be formed in order to enforce a decision. Suppose that we have six parties, whose dominance relation is as follows.



For example, d dominating a may be interpreted as: d can bend a to its will, or d can set the conditions under which cooperation will take place.

What coalitions are likely to occur? If you look at the coalition {b,c}, you may verify that they dominate the other parties: party a is dominated by b, party d is dominated by c, party e is dominated by b, while party f is dominated by b. Moreover, the coalition is internally stable, which means that they cannot directly dominate each other. One may verify that the coalition {d,e} has the same properties. These coalitions, the **von Neumann-Morgenstern stable sets**, are discussed in chapter 4.

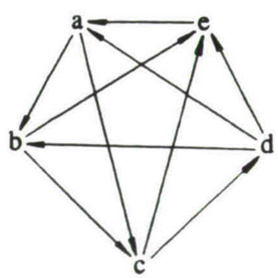
But if we take a look at the coalition {b,c}, we see that although they dominate {a,d,e,f}, they cannot reach a division of power between them that is independent of the alternative that is to be dominated. For, suppose that b and c do search for a division of power between them, (p_b,p_c) with $p_b + p_c = 1$. In order to dominate d, it is necessary that $p_c > p_b$. On the other hand, domination of e requires $p_b > p_c$. Hence, b and c cannot reach a satisfactory division of power. The same is true for {d,e}.

On the other hand, the coalition V = {b,c,d,e} is able to reach such a division of power, for example (1/4,1/4,1/4,1/4). This division has another nice feature. For each $x \in V$, we may divide $V \setminus \{x\}$ into U_x and L_x. The set U_x is formed by the alternatives that dominate x, L_x is the set of alternatives that are dominated by x. For example, $U_b = \{d\}$, $L_b = \{e\}$. The division of power (1/4,1/4,1/4,1/4) has the property that for all x the sum of power of elements in U_x is equal to the sum of power of elements in L_x. It keeps the scale of power between two clearly distinguishable groups balanced. Furthermore, everyone can tip the scale in his favour by using his own weight or power. In chapter 4, we prove that for every dominance structure, there is just one coalition that satisfies the **balanced dominant weight condition**, as described above.

This unique coalition satisfies another significant condition, the **tail stability**. If there are parties sitting down at the negotiation table, it is possible that party *a*, compared to party *b*, knows itself to be supported by a majority of external voters. Indeed, in most cases, the dominance structure between these parties is settled before the negotiations start. Now, let us denote the set of parties by X and the dominance structure by G. The assumption of the choice function *S* being tail stable means that the dominant coalition of parties in the choice set *S*(G), in our example {b,c,d,e}, can withdraw from 'the scene of battle', paying attention to more important business. They are not bothered with fights between losers in X\S(G) in the tail of the ranking $S(G) \gg X\backslash S(G)$. No matter how the comparisons between losers change, the choice set remains the same. Moreover, for a party not in $S(G)$, in order to become member of the coalition, it has to focus on the comparison with parties from the choice set. For example, it has to present better proposals to the external voters.

Example 1.4 *A choice problem*
A choice function S satisfies the **strong superset property (SSP)**, if the following holds. Let X be the set of alternatives, and let G be an asymmetric relation on X. If $S(G) \subseteq Y \subseteq X$, alternatives from X\Y may be deleted, without changing the choice set. The choice set is independent of the presence of alternatives not in Y. In other words, we may shrink the set of alternatives, without the need for an adjustment of the choice set. To give an example, the leading countries of the European Community remain leading countries, regardless of the presence of a minor, European Oriented, country.

Not every choice function fulfills the SSP. To illustrate, we consider the following tournament:



Since this tournament lacks a natural order, we look at transitive subtournaments. These are tournaments that have a (natural) best, second best, ..., worst alternative. For example, the tournaments restricted to {a,b,c}, {b,c,e}, {c,d,e} and {d,e,a} are transitive, with best elements a,b,c and d respectively. The reader may verify that these subtournaments are maximal in the following sense: they cannot be extended, without violating the assumption of transitivity. For example, the subtournament on {a,b,c} cannot be extended with d, because the tournament on {a,b,c,d} is

not transitive. The alternative e is not best element of a maximal transitive subtournament. Therefore, the so-called **Banks set** is {a,b,c,d}.
If the Banks set would satisfy SSP, we could delete the alternative e, without changing the choice set. But, as the reader may verify, this is not the case; we obtain a new Banks set equal to {a,c,d}.
In chapter 2, we will consider choice functions satisfying this independence condition.

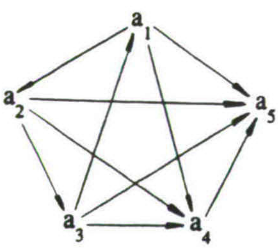
Example 1.5 *The problem of local versus global differences*
Suppose that we have five tennis players p,q,r,s and t. The results of their direct confrontations in a tournament T are listed in the matrix below.

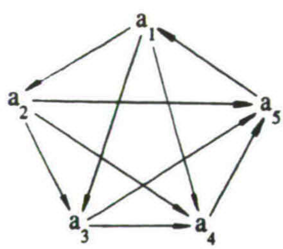
	p	q	r	s	t
p	-	1	1	1	0
q	0	-	1	1	0
r	0	0	-	1	1
s	0	0	0	-	0
t	1	1	0	1	-

Since the players p,q,r and t are on a cycle in this tournament (pTq, qTr, rTt and tTp), and each of them beats player s, we may use the ranking rule $\mathbf{f_{trans}}$, the transitive closure, to be discussed in chapter 3, to rank the players to {p,q,r,t} » {s}. Unfortunately, this rule that uses global information like whether or not two alternatives are on a cycle, violates the condition of **separability**. We can separate the set of alternatives X = {p,q,r,s,t} into B = {p,q} and X\B = {r,s,t}, such that for each alternative y from X\B the following is true. Either all alternatives from B are preferred to y, or y is preferred to each alternative in B. In a manner of speaking, the alternatives of B behave like a single alternative. Since in this case, each alternative y from X\B delivers the same local information to all b ∈ B, the local differences between b_1 and b_2 ∈ B are reduced to their local differences inside B. Therefore, a ranking rule f is said to be separable, if the relative ranking of the alternatives of B in the ranking of X, is the same as the ranking of the alternatives of B, hence without X\B. If we denote this relative ranking of B in the ranking f(T,X) of X by f(T,X)|B, this condition may be formulated as f(T,X)|B = f(T,B). If we concentrate on local differences, this may be a significant condition. Since $f_{trans}(T,B) = \{p\} \gg \{q\} \neq \{p,q\} = (\{p,q,r,t\} \gg \{s\})|B = f_{trans}(T,X)|B$, this ranking rule does not satisfy the condition of separability. On the other hand, the ranking rule $\mathbf{f_{out}}$, which orders the alternatives to the total number of comparisons in which they prevailed, leading to the ranking $\{p,t\} \gg \{q,r\} \gg \{s\}$, does satisfy this condition. But, if we concentrate on global differences, we may expect the roles to be interchanged.

We shortly discuss this condition concerning global differences, the **interval consistency**. Suppose we have a ranking of the alternatives of X, $f(T,X) = X_1 \gg X_2 \gg ... \gg X_i \gg ... \gg X_j \gg ... \gg X_n$. We may define an interval as the union of successive maximal indifference classes. In the example we choose the interval $B = X_i \cup ... \cup X_j$. Note that the alternatives from B behave like one single alternative in this ranking. Now, a ranking rule f is said to be interval consistent, if $f(T,X)|B = f(T,B)$. In other words, the relative ranking of the alternatives from B in the ranking of X is the same as the ranking of alternatives from B, hence without X\B. The rule f_{trans} satisfies this global equivalent of the separability condition. As may be deduced from the example above, f_{out} is not interval consistent, take the interval $\{p,t\}$.
In chapter 3, section 3.7, we will meet rules that satisfy both conditions.

Example 1.6 *The problem of circularity*
Consider two sets of pairwise comparisons between 5 objects $a_1,...,a_5$. We assume that a_iTa_j if $i < j$, except that in the first set a_3Ta_1 and in the second set a_5Ta_1:





Which set of comparisons is more circular, is more inconsistent?
If we look at the number of reversals needed to obtain a natural ranking from best to worst, we see that both sets require just one reversal. In the first set, this is a_3Ta_1, in the second one this must be a_5Ta_1. So, both

are equally circular with respect to this criterion.

On the other hand, the comparison a_5Ta_1 introduces more circular patterns than does a_3Ta_1. Indeed, in the second set we have three circular triads: $a_1Ta_2Ta_5Ta_1$, $a_1Ta_3Ta_5Ta_1$ and $a_1Ta_4Ta_5Ta_1$. Hence, the inconsistency a_5Ta_1 has to receive more weight than a_3Ta_1.

A great part of the discussions that one may find in literature, revolve around the justification of this weighting. If we assume that the choices in each comparison are made at random, this weighting is not justified. On the other hand, for most reasonable alternative hypotheses to randomness, the slight upset a_3Ta_1 is more likely to occur than the major upset a_5Ta_1.

In chapter 5, we will discuss two conditions that describe or formalize these two principles: weighting versus non-weighting. First of all, we have the condition of **equability**. It formalizes the idea that there does not exist a context so that some preference reversals are more sweeping and therefore must receive more weight. This condition means that the effects of all elementary changes at the local tournament level are to be weighted uniformly. Hence, the two sets of pairwise comparisons given above are equally circular. On the other hand, we have the condition of **outscore equability**. This condition relates the change of circularity that a (preference) reversal between two alternatives at the local tournament level may cause, to the difference or distance between these alternatives at a global level. Since one may view the score of an alternative as a measure of the strength or desirability of that alternative, this distance is measured using the (differences in the) scores.

Surprisingly, together with some mild assumptions, these conditions completely characterize the two wildely used circularity measures. First of all, we have **Slater's i**, which is equal to the minimal number of upsets needed to convert the tournament into a linear or strict order. This fits the principle as formalized by the equability condition. A second measure, λ, assigns to any tournament its number of **3-cycles** or circular triads. The more 3-cycles, the more circular is the tournament. As will be shown in chapter 5, it is related to the outscore equability.

1.2 DEFINITIONS AND PRELIMINARIES

In this part, we introduce some basic notions which are used throughout. Moreover, we prove some well-known theorems dealing with the relation between irreducible tournaments and the transitive closure.

1.2.1 PAIRWISE COMPARISON STRUCTURES

Throughout this thesis, the set of alternatives is taken to be finite.
We assume that the reader is familiar with the notion of a binary relation:

Binary relations

Let $U = \{u_1,...,u_n\}$ be any finite set.
A binary relation R on U is a subset of the Cartesian product $U \times U$: $R \subseteq U \times U$. If $(x,y) \in R$, we often write xRy. The relation may also be denoted by (R,U).

An element of a binary relation $R \subseteq U \times U$ is written as $(x,y) \in R$, or xRy, to be read as: x is *as least as good as* y in R, where the interpretation of 'good' depends on the situation. It may represent e.g. preference, strength or domination. If xRy and yRx, we often write $x \cong y : R$, meaning that x and y are *indifferent* in R. If $(x,y) \in R$ but $(y,x) \notin R$, we say that x is *preferred to*, is *stronger than*, *dominates*, or *beats* y.

In this monograph, we frequently restrict relations on a set of alternatives X to a subset $Y \subseteq X$. Its definition is straightforward.

Restrictions to subsets

Let (R,X) be a relation on X. If $Y \subseteq X$ is a subset of X then the restriction of R to Y is denoted by (R,Y) or $R|Y$ and is defined by: $(R,Y) = \{(x,y) \in Y^2 : (x,y) \in R\} = R \cap Y^2$.

If there is no misunderstanding about the set of alternatives, we often write R instead of (R,X),

Representation of relations

We distinguish two representations. In the *matrix representation* A of a relation R, a number 1 at position (x,y) of the matrix A means that

$(x,y) \in R$, while a number 0 means that $(x,y) \notin R$, see for instance example 1.2. The matrix A is called the *relation matrix*, or in case of tournaments (see below), the *tournament matrix*. In the matrix representation of a relation on X, we mostly put the character'-' instead of '0' on the positions A_{ii}, where $i \in \{1,...,|X|\}$.

In the *network representation* of a relation R, an arrow from x to y means that $(x,y) \in R$. If there is no arrow from x to y then $(x,y) \notin R$. Compare figure 1.2.

Properties for binary relations

A binary relation R on U is

(a) *reflexive, iff*[1] *for all $u \in U$: $(u,u) \in R$.*

(b) *antisymmetric, iff for all $(x,y) \in U \times U$:*

if $(x,y) \in R$ and $(y,x) \in R$ then $x = y$.

(c) *asymmetric, iff for all $(x,y) \in U \times U$:*

if $(x,y) \in R$ then $(y,x) \notin R$.

(d) *complete, iff for all $(x,y) \in U \times U$ with $x \neq y$, $(x,y) \in R$ or $(y,x) \in R$.*

(e) *transitive, iff for all $(x,y), (y,z) \in R$: $(x,z) \in R$.*

In chapter 2, 3 and 5, we restrict our attention to tournaments. This name stems from real-life round-robin tournaments: each alternative is compared with each other alternative. In all these pairwise comparisons, a strict choice for one of them is made.

Tournaments

A binary relation R on U is a tournament iff R is complete and asymmetric.

We introduce two sets of relations on U, the set of asymmetric relations and the set of tournaments.

Definition of $\mathbb{G}(X)$ and $\mathbb{T}(X)$

Let $U = \{u_1,...,u_n\}$ be a finite set.

(a) *If $\emptyset \neq X \subseteq U$ then $\mathbb{G}(X)$ is the set of all asymmetric relations on X.*

(b) *If $\emptyset \neq X \subseteq U$ then $\mathbb{T}(X)$ is the set of all complete asymmetric relations on X.*

[1] if and only if

Note that $\mathbb{T}(X) \subseteq \mathbb{G}(X)$.

Linear orders and weak orders

Let $U = \{u_1,...,u_n\}$ be a finite set.

(a) Let $\emptyset \neq X \subseteq U$. A reflexive, complete and transitive relation on X is called a weak order on X. $\mathbb{W}(X)$ is the set of weak orders on X.

(b) Let $\emptyset \neq X \subseteq U$. A reflexive, complete, antisymmetric and transitive relation on X is called a linear order on X. $\mathbb{L}(X)$ is the set of linear orders on X.

Note that $\mathbb{L}(X) \subseteq \mathbb{W}(X)$, for all nonempty $X \subseteq U$, $\mathbb{L}(X) \neq \mathbb{W}(X)$ if $|X| \geq 2$.

Representation of weak orders

In general, a weak order R on X may be written as $X_1 \gg X_2 \gg ... \gg X_k$ for some $k \in \mathbb{N}$, where $X_1, X_2, ..., X_k$ is a partition of X. In this notation every X_i is a maximal indifference class (each alternative in X_i is indifferent to every other alternative in X_i, maximality means that for every element x in the complement of X_i, there is an element y in X_i such that x and y are not indifferent), and $X_i \gg X_j$ means that each alternative in X_i is preferred to or ranked ahead of each alternative in X_j. So, each weak order may be seen as a strict order of its maximal indifference classes. A weak order is a linear order if each indifference class consists of only one alternative, compare the linear order in example 1.2.

For two alternatives $a,b \in X$, let $a \in X_i$, $b \in X_j$. We often write $a \gg b : R$ if $j > i$, $a \geq b : R$ if $j \geq i$ and $a \cong b : R$ if $i = j$.

1.2.2 ELEMENTARY OPERATIONS ON BINARY RELATIONS

Throughout the chapters, we make use of some elementary operations on binary relations. We give their definitions and discuss them shortly.

Transitive closure

Let $G \in \mathbb{G}(X)$ and $Y \subseteq X$. Then $\tau(G,Y) \subseteq Y^2$, the transitive closure of (G,Y), is defined by:

for all $x,y \in Y$, we have $x(\tau(G,Y))y$ iff

$x = y$, or there exist alternatives $z_0,...,z_k \in Y$, $z_0 = x$, $z_k = y$, such that z_iGz_{i+1}, $i \in \{0,...,k-1\}$.

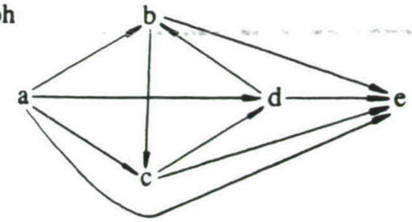
So, $x(\tau(G,Y))y$, if $x = y$ or there exists a path in (G,Y) with x as its starting point and y as its end. For each complete relation G, τG is a weak order.

Example 1.7

Take $X = \{a,...,e\}$ and T_2 equal to

T_2	a	b	c	d	e
a	-	1	1	1	1
b	0	-	1	0	1
c	0	0	-	1	1
d	0	1	0	-	1
e	0	0	0	0	-

with graph

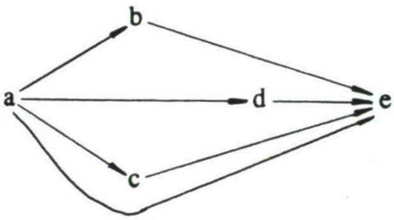


Then $\tau T_2 = \{a\} \gg \{b,c,d\} \gg \{e\}$; a weak order.

Asymmetric part

Let R be a binary relation on X. Then the asymmetric part of R, denoted by aR (or $a(R,X)$) is defined by: $aR = \{(x,y) \in R: (y,x) \notin R\}$.

Compared with the original relation G, the asymmetric part does not contain indifferences. If we let G be equal to τT_2, we obtain the following graph for $a\tau T_2$:



Permutation

Let σ be a permutation on X; $\sigma : X \to X$ is a bijection. For all relations R on X, σR is a relation on X defined by:

$(\sigma(x),\sigma(y)) \in \sigma R$ iff $(x,y) \in R$.

Take $R = T_2$ from example 1.7, and define $\sigma(a) = a$, $\sigma(b) = c$, $\sigma(c) = d$, $\sigma(d) = b$ and $\sigma(e) = e$. Then $\sigma T_2 = T_2$.

Maximal and best elements

Let R be a binary relation on $X \subseteq U$. Then
$Max(R,X) = \{x \in X$: *there is no* $y \in X$ *such that* $y(aR)x\}$.
$Best(R,X) = \{x \in X$: *for all* $y \in X,\ y \neq x,\ xRy\}$.

Hence $x \in Max(R,X)$, if x is not strictly beaten by another alternative. Further, $x \in Best(R,X)$, if x beats all other alternatives. Note that $Best(R,X) \subseteq Max(R,X)$.
Let T be the tournament of example 1.2. Then $Max(T) = Best(T) = \emptyset$, while $Max(\tau T) = \{a,b,c,d,e\}$. In the example above, $Max(T_2) = Best(T_2) = \{a\}$.

An important operation is the concatenation of two or more relations.

Concatenation

Let $X = Y \cup Z \subseteq U$, $Y \cap Z = \emptyset$. *We define the concatenation* $(G,Y) \gg (G,Z) \in \mathbb{G}(X)$ *of* $(G,Y) \in \mathbb{G}(Y)$ *and* $(G,Z) \in \mathbb{G}(Z)$ *by:*
$(G,Y) \gg (G,Z) :=$

$$\{(u,v) \in X\times X : (u,v) \in (G,Y) \text{ or } (u,v) \in (G,Z) \text{ or } u \in Y,\ v \in Z\}.$$

Hence, each alternative of Y is preferred to every alternative of Z. Note that the tournament of example 1.7 is a concatenation of three tournaments: $T_2 = (T_2,\{a\}) \gg (T_2,\{b,c,d\}) \gg (T_2,\{e\})$, where we make use of the associativity of the concatenation operation.

1.2.3 PRELIMINARIES

In this part, we consider the relation between irreducible tournaments and cycles. Furthermore, we show how to compute the transitive closure.

First of all, we introduce the notion of a cycle.

Cycles

Let G be an asymmetric relation on X. A cycle of length k is a sequence of distinct alternatives $\{x_1,\dots,x_k\} \subseteq X$, *such that* $x_1Gx_2G\dots Gx_kGx_1$.

A special type of cycle, is the Hamilton cycle. It is a cycle that 'visits' each alternative precisely once.

Hamilton cycle

A Hamilton cycle of an asymmetric relation $G \in \mathbb{G}(X)$ is a cycle of length n, where $n = |X|$.

In the tournament T_2 of example 1.7, we have one cycle: $bT_2cT_2dT_2b$. A Hamilton cycle for the tournament T of example 1.2 is given by aTbTcTdTeTa.

Irreducible parts of a tournament

As is well-known, an asymmetric and complete relation can be decomposed into irreducible parts. See e.g. Moon (1968). We introduce the notion of irreducible tournaments, and give an alternative proof for this well-known fact.

A tournament (T,Y) is irreducible, iff for each partition $Y = A \cup B$, such that $A \cap B = \emptyset$, we have: $(T,Y) \neq (T,A) \gg (T,B)$.

So, a tournament T is irreducible if it is not possible to partition its alternatives into two nonempty sets A and B such that for all pairs of alternatives $(a,b) \in A \times B$, we have $(a,b) \in T$.
To give an example, consider again the circular tournament T_1. As one may easily verify $T_1 \neq (T_1,\{a,b\}) \gg (T_1,\{c,d\})$, in fact it is irreducible. On the other hand, a linear tournament on two or more alternatives is reducible.

THEOREM 1.1 *Each tournament (T,X) can uniquely be decomposed into a concatenation of irreducible tournaments.*

Proof (Existence) If (T,X) is irreducible, the existence is evident. If (T,X) is reducible then $(T,X) = (T,A) \gg (T,B)$ for some A and B. If (T,A) and (T,B) are irreducible, then (T,X) is the concatenation of these irreducible tournaments. Otherwise, apply the same line of reasoning to (T,A) and/or (T,B).
(Unicity) Suppose that $(T,A_1) \gg \dots \gg (T,A_k) = (T,B_1) \gg \dots \gg (T,B_m)$ are concatenations of irreducible tournaments. Since both A_1 and B_1 are preferred to all other alternatives, $C_1 := A_1 \cap B_1 \neq \emptyset$. Suppose that C_1 is strictly contained in either A_1 or B_1. This means that all alternatives from C_1 are preferred to all other alternatives, which contradicts the

irreducibility of either (T,A_1) or (T,B_1). So, $A_1 = B_1$. Similarly, we obtain $A_2 = B_2, ..., A_k = B_k$. ■

This theorem teaches us that each tournament $(T,X) \in \mathbb{T}(X)$ may uniquely be decomposed into $(T,X) = (T,X_1) \gg (T,X_2) \gg ... \gg (T,X_n)$ where the tournaments (T,X_i), $i \in \{1,...,n\}$ are the so-called irreducible parts of (T,X). Take $X = \{a,...,e\}$ and T_2 equal to the tournament in example 1.7. Then $(T_2,X) = (T_2,\{a\}) \gg (T_2,\{b,c,d\}) \gg (T_2,\{e\})$. Note that the theorem also holds for arbitrary asymmetric relations.

There is a connection between irreducible tournaments and the occurrence of Hamilton cycles, which is expressed in the following theorem, see Moon (1968).

THEOREM 1.2 *Let T be a tournament. Then T is irreducible iff T has an Hamilton cycle.*

Proof (only if) Let T be an irreducible tournament on X with $|X| = n$. We construct a Hamilton cycle as follows. Since T is irreducible, there is at least one cycle, so we assume the existence of a cycle $C \subseteq X$, of length $k < n$:

$$x_1Tx_2T...Tx_kTx_1.$$

We show that we may extend C to a Hamilton cycle.
Suppose that we are able to choose an element $a \in X\backslash C$, such that there is an $i \in \{1,...,k\}$ with x_iTa and aTx_{i+1}. Then we may extend the cycle C to $x_iTaTx_{i+1}T...Tx_{i-1}Tx_i$.
If we cannot find such an element a, then we may divide $X\backslash C$ into two subsets W and S such that for all $w \in W$ and $s \in S$ we have x_iTw and sTx_i, for all $i \in \{1,...,k\}$. Since T is irreducible, W and S are nonempty and there exist $w' \in W$ and $s' \in S$ such that $w'Ts'$. But then, we may extend C to the cycle $x_1Tw'Ts'Tx_2T...Tx_kTx_1$.
So, we are able to extend C to a cycle of length k+1 or k+2. Continuing in this manner, we finally obtain a cycle of length n, a Hamilton cycle.
(if) Evident. ■

Together with theorem 1.1, this implies that if $(T,X_1) \gg ... (T,X_n)$ is the decomposition of (T,X) into irreducible parts, the transitive closure of

(T,X) is equal to the weak order $X_1 \gg \dots \gg X_n$.

To give an impression, consider figure 1.3, where it is to be understood that if an arrow has not been drawn, it is oriented from left to right. The cycles are Hamilton cycles for the irreducible subtournaments (T,X_i), where $i \in \{1,\dots,n\}$.

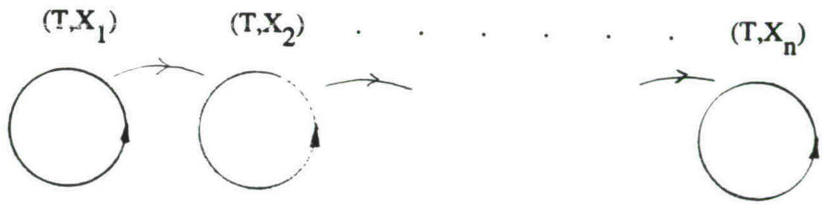


Figure 1.3 Illustration of the decomposition into irreducible parts.

An important tool to measure the performance of a player in a tournament, is the score. In graph-theoretic terms it is the out-degree.

Copeland scores in tournaments

Let T be a tournament on X. Then the (Copeland) score of alternative $x \in X$ *is equal to* $s_x(T) = |\{y \in X : xTy\}|$, *(Copeland, 1951). If the choice of tournament is clear, we often write* s_x.
The score vector is $\underline{s} = (s_x(T))_{x \in X}$, *and is equal to* $A\underline{e}$, *A being the tournament matrix and* $\underline{e}$ *a vector of 1's.*

So, the score of an alternative x is equal to the number of alternatives that are preferred by x.

Let T be a tournament on X, $n = |X|$. Suppose that $\underline{s}$ is its score vector, such that $s_i \geq s_j$ if $i \geq j$. Then

(a) $\sum_{i=1}^{n} s_i = \binom{n}{2}$

(b) $\sum_{i=1}^{k} s_i \geq \binom{k}{2}$, $k \in \{1,\dots,n\}$.

Indeed, the number of pairwise comparisons in a tournament on n alternatives is equal to $\binom{n}{2}$, which proves (1). The sum of Copeland scores of the tournament restricted to $\{a_1,\dots,a_k\}$, is equal to $\binom{k}{2}$. But in the sum presented in (2), also comparisons with alternatives belonging to the last n-k alternatives are counted, which proves (2).

As for the computation of irreducible parts, we present proposition 1.3. It may be found by combining page 2 and 61 from Moon (1968). Note that for all $k > 0$, $\binom{k}{2} = 0 + 1 + 2 + \ldots + k-1$.

PROPOSITION 1.3 *Let T be a tournament on X, $n = |X|$. Suppose that $\underline{s}$ is its score vector, such that $s_i \geq s_j$ if $i \geq j$. We denote the corresponding alternatives with $a_1,\ldots,a_n$.*
The alternatives $a_{h+1},\ldots,a_j$ constitute an indifference class of the transitive closure of T, and hence an irreducible part of T iff

(1) $\sum_{i=1}^{h} s_i = \binom{h}{2}$ *or* $h = 0$,

(2) $\sum_{i=1}^{j} s_i = \binom{j}{2}$,

(3) *if* $h+1 < j$ *then* $\sum_{i=1}^{k} s_i > \binom{k}{2}$, *for* $k \in \{h+1,\ldots,j-1\}$.

Proof Let us suppose that $(x,y) \in a\tau T$, the asymmetric part of the transitive closure of T. Then, using figure 1.3, we may deduce that $s_x > s_y$. Indeed, we have xTy, and if yTz then xTz. Hence, an indifference class of τT is equal to $\{a_u,\ldots,a_{u+v}\}$ for some u and v such that $1 \leq u$ and $u+v \leq n$. Second, observe that a section between the alternatives $A = \{a_1,\ldots,a_j\}$ and $B = \{a_{j+1},\ldots,a_n\}$ separates indifference classes of τT, if for all elements $x \in B$ and all elements $y \in A$, we have that not $(y,x) \in T$, so that we obtain $\sum_{i=1}^{j} s_i = \binom{j}{2}$. ■

Example 1.8
Take the tournament T_2 of example 1.7. Then, as one may easily verify, the score vector $\underline{s}$ is equal to $(0,2,2,2,4) = (s_e,s_b,s_c,s_d,s_a)$. Since $s_e = 0$, $\{e\}$ constitutes an irreducible part of T_2 and the last indifference class of τT_1. Because
$0 + 2 > 0 + 1$ and $0 + 2 + 2 > 0 + 1 + 2$ and $0 + 2 + 2 + 2 = 0 + 1 + 2 + 3$, $\{b,c,d\}$ is the second indifference class of τT_2. The first indifference class is $\{a\}$: $0 + 2 + 2 + 2 + 4 = 0 + 1 + 2 + 3 + 4$.

APPENDIX: INDEX OF PROCEDURES AND CONDITIONS

In the tables, the numbers correspond to the page numbers.

◊ CHOICE FUNCTIONS

Table I: Choice functions

Top cycle, *tc*	ch. 2: 27	
Uncovered set, *uc*	ch. 2: 29	ch. 4: 119
Minimal covering set, *mc*	ch. 2: 31	
Banks set, *b*	ch. 2: 35	
Strategy equilibrium, *strat*	ch. 2: 45	ch. 4: 119, 134
Copeland choice set, *cop*	ch. 2: 38	
Slater choice set, *sl*	ch. 2: 39	
T-equilibrium set, *teq*	ch. 2: 36	
Von Neumann-Morgenstern, *vNM*		ch. 4: 117, 137
Maximal elements, *top*		ch. 4: 119

◊ RANKING RULES

Table II: Ranking rules

f_{trans}	ch. 3: 67	f_{eigen}	ch. 3: 97
f_{out}	ch. 3: 70	f_{slat}	ch. 3: 87
f^*_{out}	ch. 3: 72	f_{vNM}	ch. 4: 141
f_{cop}	ch. 3: 71	f_{top}	ch. 4: 141
f_{prob}	ch. 3: 94	f_{strat}	ch. 4: 142

• CRITERIA FOR CHOICE FUNCTIONS AND RANKING RULES

In this monograph, we introduce and discuss lots of intuitively appealing conditions for ranking and choice methods. In table III, we list them. Also compare the tables at the end of each chapter. Note that for conditions mentioned in table III^b, the precise formulation for ranking rules and choice functions is of course different.

Table III[a]: Conditions for choice functions

Condorcet properties	ch. 2: 26	ch. 4: 122
Concatenation consistency	ch. 2: 26	
Neutrality	ch. 2: 26	
γ-property	ch. 2: 32	
$\hat{\gamma}$-property	ch. 2: 33	
Strong superset property, SSP	ch. 2: 33	ch. 4: 138
Monotonicity	ch. 2: 34	
Sen's condition	ch. 2: 34	
Chernoff's condition	ch. 2: 51	
Balanced dominant weights	ch. 2: 46	ch. 4: 132
Local differences		ch. 4: 123

Table III[b]: Conditions for choice functions and ranking rules

Δ-IIA	ch. 2: 48	ch. 3: 78	ch. 4: 138
Separability	ch. 2: 53	ch. 3: 90	
Global monotonicity	ch. 2: 57		ch. 4: 138
Tail stability	ch. 2: 59		ch. 4: 121, 142
Independence of 3-cycle orientation		ch. 3: 81	
External stability			ch. 4: 116,144

Table III[c]: Conditions for ranking rules

Commutation properties	ch. 3: 64, 73	
Free of upsets	ch. 3: 68	
Right-interval consistency	ch. 3: 66	
Weakly-Δ-IIA	ch. 3: 80	
Interval consistency	ch. 3: 90	ch. 4: 142
Partial independence	ch. 3: 73	
Independence of interval changes	ch. 3: 69	
Positive responsiveness	ch. 3: 85	

◊ CIRCULARITY MEASURES

In the following two tables, we list the measures and conditions.

Table IV: Circularity measures

• CRITERIA FOR CIRCULARITY MEASURES

Table V: Conditions for circularity measures

CHAPTER 2

CHOICE FUNCTIONS ON PREFERENCE RELATIONS

2.1 INTRODUCTION

Given is a finite set $U = \{u_1,...,u_n\}$ of alternatives, and a tournament T on $X \subseteq U$, which expresses decisive judgments for all pairs of alternatives. It is well-known that often tournaments do not contain a best alternative. Therefore, we construct and compare choice functions, that handle the difficulty posed by the nonexistence of a clear winner in different ways.

Let $X \subseteq U$ be a nonempty set. Then $\mathbf{T}(X)$ is the set of tournaments on X. In this chapter, we often will use variable sets of alternatives. Therefore, we introduce the set $\mathbf{T}$, which is the union of all sets $\mathbf{T}(X)$, where $X \subseteq U$ is a nonempty set. Analogously to $\mathbf{T}$, the set $\mathbf{G}$ is the union of all sets of asymmetric relations $\mathbf{G}(X)$ on X, where $\varnothing \neq X \subseteq U$. Note that if $(G,X) \in \mathbf{G}(X)$ and $Y \subseteq X$, then (G,Y), the restriction of G to Y, is element of $\mathbf{G}(Y)$.

Choice functions

A choice function S on $\mathbf{G}(X)$ is a function $S : \mathbf{G}(X) \to 2^X \backslash \{\varnothing\}$. A choice function S on $\mathbf{G}$ is a family of choice functions, one for each $\mathbf{G}(X)$, $\varnothing \neq X \subseteq U$. Completely analogous, we may define choice functions on $\mathbf{T}(X)$ and $\mathbf{T}$.

Most conditions that we will consider, are stated in terms of asymmetric relations. But, of course, they may be transformed to conditions stated in terms of tournaments, by changing $\mathbf{G}$ to $\mathbf{T}$.

2.2 ELEMENTARY PROPERTIES

We provide three elementary conditions. Throughout this chapter, we only consider choice functions for which these conditions hold.

- **Condorcet consistency** (see e.g. Moulin, 1986)

  A choice function S on $\mathbb{G}(X)$ satisfies the condition of Condorcet consistency if for all $(G,X) \in \mathbb{G}(X)$ and all $a \in X$ we have that if aGb for all $b \in X \setminus \{a\}$ then $S(G,X) = \{a\}$.

For tournaments, this condition means that if a tournament has a best element, this unique element constitutes the choice set.

We next introduce another consistency condition.

- **Concatenation consistency** (compare e.g. Storcken, 1989)

  A choice function S on $\mathbb{G}$ satisfies the condition of concatenation consistency if for all $A,B \subseteq U$ such that $A \cap B = \varnothing$ and all $G \in \mathbb{G}(A \cup B)$:

  $S[(G,A) \gg (G,B)] = S(G,A)$.

As for the concatenation consistency: if each alternative from B is defeated by all alternatives from A, the choice set from $(G,A) \gg (G,B)$ only depends upon (G,A). Note that the Condorcet consistency is implied by the concatenation consistency.

Let σ be a permutation on X. If G is an asymmetric relation on X, then $\sigma G \in \mathbb{G}(X)$ is defined by: $(\sigma x, \sigma y) \in \sigma G$ iff $(x,y) \in G$.

- **Neutrality** (see e.g. Moulin, 1986)

  A choice function S on $\mathbb{G}(X)$ is neutral if for permutations σ of X and all $(G,X) \in \mathbb{G}(X)$, we have $S(\sigma(G,X)) = \sigma S(G,X)$.

Neutrality excludes the possibility that a choice function S makes a distinction between different ways of naming the alternatives. Compare the commutation with permutation, discussed in chapter 3, section 2.

Let S be a choice function on $\mathbb{G}$. If for each $\varnothing \neq X \subseteq U$, $S : \mathbb{G}(X) \to 2^X \setminus \{\varnothing\}$ satisfies a certain property, say ϕ, then S is said to satisfy property ϕ.

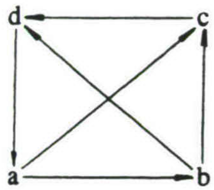
For example, a choice function S on $\mathbb{G}$ is neutral if each member of the family of choice function $S : \mathbb{G}(X) \to 2^X \setminus \{\varnothing\}$, $\varnothing \neq X \subseteq U$, is neutral.

2.3 CHARACTERIZATION OF THE TOP CYCLE

In this section, we give a characterization of the choice function tc, which assigns to a tournament its top cycle.

An alternative u is in the top cycle of a tournament $T \in \mathbb{T}(X)$, if for all $x \in X$, $u(\tau T)x$, where τT is the transitive closure of T, see chapter 1, section 1.2.2. Unfortunately, the top cycle often is unmanageably large. This may be seen by taking a linear order and reversing the order between top and bottom element. But it has another drawback. If, for example, the tournament stems from pairwise majority voting, the top cycle may contain Pareto dominated alternatives. To give an example, consider a cyclic tournament on 4 alternatives as depicted below.

T_1



It is possible that all agents vote for alternative b in the comparison between alternative b and c. So bT_1c is unanimous. For example, this occurs when we have three voters with the following individual preference:

a » b » c » d

d » a » b » c

b » c » d » a.

Nevertheless, using the choice function *tc*, together with a,b and d, the Pareto dominated alternative c is chosen.

To avoid choosing Pareto dominated alternatives when the tournament stems from pairwise majority voting, we will introduce other choice functions in the sections to come.

But first, we provide a characterization for the choice function *tc*. For this purpose, we consider the following condition.

- **Condorcet transitivity** (see e.g. Moulin, 1986)

  A choice function $S : \mathbb{G}(X) \rightarrow 2^X \setminus \{\emptyset\}$ *satisfies the Condorcet transitivity if:*

  for all $G \in \mathbb{G}(X)$ *and all* $a,b \in X$: *if* $a \in S(G)$ *and* bGa *then* $b \in S(G)$.

This condition means that alternatives from the top cycle have to be chosen, see theorem 2.1.

Smallest choice function

In the following theorem, we use the term 'smallest' choice function. A choice function S is smaller than a choice function S', if for all tournaments T, we have that $S(T) \subseteq S'(T)$, and $S \neq S'$.

THEOREM 2.1 (Characterization of the top cycle)

(a) The choice function tc is the smallest choice function on $\mathbb{T}$, *that satisfies the Condorcet transitivity.*

(b) If a choice function S satisfies the Condorcet transitivity and the concatenation consistency, then $S = tc$.

Proof (a) Let $T \in \mathbb{T}(X)$. Suppose that $b \in S(T)$. For all $a \in tc(T)$, there exists a path along T from a to b: $a = x_0Tx_1T...Tx_k = b$, $k \geq 0$. But then, applying the Condorcet transitivity property k times, we conclude that $a \in S(T)$. (b) Left to the reader. ■

We will meet other theorems that look alike to the characterization given above. These kinds of characterizations make clear the balancing between certain consistency conditions, in this case the Condorcet transitivity, and the desire to obtain small choice sets. In general, the conditions give a lower bound to the size of the choice set. If we relax the conditions, more choice sets will satisfy these conditions, giving rise to a reduction of the lower bound.

If we do not consider the top cycle of a tournament as a candidate for the final choice, because in general it is too large, theorem 2.1 implies that we have to relax the assumption of Condorcet transitivity.

In the following section, we will discuss another characterization of *tc*.

The difference with the characterization given there, is that in section 2.4.2, we use independence conditions in terms of variable sets of alternatives.

2.4 THE UNCOVERED SET AND THE MINIMAL COVERING SET

In this section, we introduce and discuss some choice functions that resolve the problem of choosing Pareto dominated alternatives. We introduce the uncovered set and the minimal covering set. In the second subsection, we look at some properties and characterizations for these sets and the top cycle.

2.4.1 The choice sets uc, uc* and mc

The uncovered set: uc

In Fishburn (1977) and Miller (1980) the cover relation was introduced. Given an asymmetric relation G, it is defined as:

x covers y iff

xGy and for all $z \in X$: if yGz then xGz.

We write xCy if x covers y.

The uncovered set of G, denoted by *uc*(G) is the set of maximal elements in this cover relation. See chapter 1 for the definition of maximal elements. Since the cover relation C is transitive and X is finite, *uc*(G) is not empty.

Note that for each tournament T originating from pairwise majority voting, *uc*(T) only contains Pareto undominated alternatives. To show this, recall that all parties involved in the majority voting procedure are supposed to have a transitive preference over the existing alternatives. Now, let $x \in X$ and suppose that the alternative y is unanimously preferred to x in the tournament T. Then, if xTz, meaning that the majority prefers x to z, the same majority prefers y to z, implying yCx. Hence $x \notin$ *uc*(T).

In case of majority voting one is interested only in uncovered proposals.

Indeed, suppose that xCy, then it is not very attractive to choose y: y is overruled by x.
The computation of *uc*(T) is straightforward if we use the following proposition. Let A be the tournament matrix, see chapter 1.

PROPOSITION 2.2 (Miller, 1980) *Let $T \in \mathbb{T}(X)$ be a tournament. The following three statements are equivalent. Let $B = A + I$.*

(1) $x \in X$ *is uncovered*

(2) *for all* $y \neq x$ *in* X, *there is a path along* T *from* x *to* y *of length 1 or 2*

(3) *for all* $y \in X$; $(B^2)_{xy} \neq 0$.

Proof ($1 \Rightarrow 2$) Suppose that yTx. Since x is uncovered, there is an element z, such that xTz and zTy, meaning that x can reach y in two steps. ($2 \Rightarrow 3$). The entry $(B^2)_{xy}$ is equal to the number of paths of length 1 or 2 from x to y, hence it is not equal to 0. ($3 \Rightarrow 1$). The alternative x is able to reach any other alternative in 1 or 2 steps. So yCx for any y is impossible. ■

The uncovered set arose in the study of majority voting. Most of the choice functions which we will introduce hereafter, were originally inspired by the theory of collective choice, in particular the strategic analysis of voting rules based upon majority voting.
In this light, the uncovered set itself does have a shortcoming. For example, there may be domination within the uncovered set. To give such an example, consider (T',X), X = {a,b,c,d,e}

T'	a	b	c	d	e
a	-	1	0	1	1
b	0	-	1	1	0
c	1	0	-	0	1
d	0	0	1	-	1
e	0	1	0	0	-

uc(T',X) = {a,b,c,d}, while *uc*(T',{a,b,c,d}) = {a,b,c}.

Therefore, one tries to find refinements of the choice function *uc*. We consider the recursive uncovered set and the minimal covering set.

Composition of choice functions

Let S, S' be two choice functions on $\mathbb{G}$*. Then the composition* $S \circ S'$ *is a choice function on* $\mathbb{G}$ *defined by* $S \circ S'(G,X) = S(G,S'(G,X))$*, for all relations* $(G,X) \in \mathbb{G}$.

The recursive uncovered set: uc^*

Let uc^* be the recursive application of *uc*, defined as follows.
We introduce a sequence of choice functions on $\mathbb{G}$,

$uc_1, uc_2, \ldots$, defined by

$$uc_1 = uc \text{ and if } k \geq 2 \text{ then } uc_k = uc \circ uc_{k-1}.$$

Now, since $\mathbb{G}$ is finite, there does exist an integer α, such that $uc_k = uc_\alpha$ for all $k \geq \alpha$.

We define $uc^* = uc_\alpha$.

It is clear that the relation G restricted to $uc^*(G)$ is irreducible for each asymmetric relation $G \in \mathbb{G}$.

The minimal covering set: mc

In Dutta (1988) a new choice function on $\mathbb{T}$ was introduced.
Let $T \in \mathbb{T}(X)$. A set $D \subseteq X$ is a covering set of the tournament T iff

(i) $uc(T,D) = D$ and

(ii) if $x \in X \setminus D$ then $x \notin uc(T,D \cup \{x\})$.

Hence, a subset D is a covering set, if all elements from D are uncovered in (T,D) and, moreover, each alternative x from $X \setminus D$ is covered in $(T,D \cup \{x\})$.

In Dutta (1988) it is proved that $uc^*(T)$ is a covering set. Moreover, he showed that $mc(T) := \cap\{D\text{: D is a covering set of T}\}$ is a covering set of T. It is called the minimal covering set.

It is contained in the top cycle of *uc*(T): $mc(T) \subseteq tc \circ uc(T)$. Moreover, it is contained in $uc^*(T)$.

We now present table 2.1, containing the choice sets of the choice functions introduced so far, for the tournament T_2.

T_2	a	b	c	d	e	f	g	h
a	-	1	1	1	1	1	0	0
b	0	-	1	1	1	1	0	1
c	0	0	-	1	1	1	1	1
d	0	0	0	-	1	0	1	1
e	0	0	0	0	-	1	1	1
f	0	0	0	1	0	-	1	0
g	1	1	0	0	0	0	-	1
h	1	0	0	0	0	1	0	-

tc: {a,b,c,d,e,f,g,h}
uc: {a,b,c,g,h}
uc^*: {a,b,c,g}
mc: {a,c,g}

Table 2.1 Choice sets of the tournament T_2.

2.4.2 Characterizations for uc, mc and tc.

In this subsection, we consider properties that are in terms of variable domains.

We start with a property that is used in a characterization of *uc*.

- γ: (Sen, 1977)
  A choice function S on $\mathbb{G}$ satisfies the γ property if for all nonempty $X,X' \subseteq U$, for all $G \in \mathbb{G}(U)$: $S(G,X) \cap S(G,X') \subseteq S(G,X \cup X')$.

This condition means the following. Suppose that $a \in X \cap X'$ is chosen when only elements from X are available. Moreover, it is chosen when only elements from X' are available. Then a has to be chosen from the union of X and X'. Stated otherwise, joining forces against a does not pay. This condition sometimes is called the expansion condition. If alternative a is able to cover the whole set U by suitably chosen subsets B_i such that $a \in S(G,B_i)$ for each i, then $a \in S(G,U)$.

If we combine the consistency conditions, we obtain the following characterization:

THEOREM 2.3 (Moulin 1986, characterization of the uncovered set). *The choice function uc on $\mathbb{T}$ is the smallest choice function that satisfies neutrality, γ and Condorcet consistency.*

Proof We give a sketch of the proof. Let a choice function S satisfy the conditions mentioned above. Take $b \in uc(T,X)$. We prove that $b \in S(T,X)$. Divide $X \backslash \{b\}$ into $X^+ = \{x \in X: bTx\}$ and $X^- = \{x \in X: xTb\}$. Because of Condorcet consistency, we have that $\{b\} = S(T,\{b,x\})$, for all $x \in X^+$. Because $b \in uc(T)$, all elements from X^- are reachable in two steps. Thus for all $x \in X^-$, there does exist an element $y \in X^+$ such that $(T,\{x,b,y\})$ is a 3-cycle. Hence, because of neutrality, for all $x \in X^-$, $b \in S(T,\{x,b,y\})$. Using the property γ, we know that $b \in S(T)$. ■

In order to be able to present a characterization of *mc*, we introduce a few more conditions.

First of all, we have a weaker version of the γ-condition.

- $\hat{\gamma}$: (See e.g. Dutta, 1988)

  Let $G \in \mathbb{G}(U)$. Suppose $X_1,\dots,X_k$ is a finite collection of subsets of U. Then

  $a \in \bigcap_{i=1}^{k} S(G,X_i)$ implies $\bigcup_{i=1}^{k} X_i - \{a\} \neq S(G, \bigcup_{i=1}^{k} X_i)$.

Note that if a choice function satisfies γ, then it also satisfies $\hat{\gamma}$.

The following condition is an independence condition.

- **Strong superset property** (See e.g. Dutta, 1988)

  A choice function S on $\mathbb{G}$ satisfies the strong superset property (SSP) if for all $G \in \mathbb{G}(U)$ and for all $X,X' \subseteq U$:

  if $S(G,X') \subseteq X \subseteq X'$ then $S(G,X) = S(G,X')$.

To describe SSP in words: if $S(G,X') \subseteq X \subseteq X'$, elements outside X may be deleted, without changing the choice set. It is independent of the presence of elements outside $S(G,X')$. The condition SSP is a strong one, most choice functions do not satisfy it. Nevertheless, it may be significant in certain circumstances. If just before the announcement of the name of the champion one of the losers is disqualified, we do not want that this suddenly leads to another champion. To give another example, the leading countries of the European Community remain leading countries, regardless of the presence of a minor, European oriented, country.

Let $X_x = X \setminus \{x\}$.

- **Monotonicity** (See e.g. Dutta, 1988)

  A choice function S on $\mathbb{G}(X)$ is monotone iff for all $G, G' \in \mathbb{G}(X)$ and all $x \in S(G,X)$,

  if for all $b \in X$ xGb implies $xG'b$, and $G|X_x = G'|X_x$ then $x \in S(G',X)$.

So, if at the binary relation level, nothing changes at the expense of x, the same is true in the final choice at the more global level of analysis.

Putting these conditions together, we obtain

THEOREM 2.4 (Dutta, 1988, characterization of the minimal covering set). *The choice function mc on $\mathbb{T}$ is the smallest choice function that satisfies monotonicity, neutrality, SSP and $\hat{\gamma}$.* ■

One of the characterizing properties for $S = tc$ (top cycle), using variable domains is:

- **Sen's condition** (See e.g. Moulin, 1986)

  A choice function S on $\mathbb{G}$ satisfies Sen's condition if for all $G \in \mathbb{G}(U)$ and all $Y \subseteq X$:

  if $S(G,Y) \cap S(G,X) \neq \emptyset$ then $S(G,Y) \subseteq S(G,X)$.

Let a choice function S satisfy this condition. If we extend the set of outcomes, then, if one of the old champions is chosen, all of them are chosen.

THEOREM 2.5 (Moulin 1986, characterization of the top cycle). *The choice function tc on $\mathbb{T}$ is the smallest choice function that satisfies neutrality, γ, Condorcet consistency and Sen's condition.* ■

2.5 OTHER CHOICE SETS

In this section, we discuss the so-called Banks set of a tournament, the tournament equilibrium set, the Copeland choice function and the Slater choice set.

2.5.1 The Banks set

In Banks (1985) the now so-called Banks set was introduced. Given a tournament T, the Banks set $b(T)$ is defined to be the set of alternatives which are top element of at least one transitive subtournament of T, maximal with respect to inclusion. Such subtournaments may be seen as stable hierarchies; they cannot be extended without violating this stability. In the tournament of table 2.1, such a subtournament is given by hT_2aT_2f. Indeed, it cannot be extended, without violating the transitivity, indicating that $h \in b(T_2)$. The reader is invited to verify that $b(T_2) = \{a,b,c,g,h\}$.

The motivation for this choice function is the following.

Consider a group of voters, faced with a set of alternatives. As already mentioned, if the tournament stems from pairwise majority voting, the top cycle sometimes does contain Pareto dominated alternatives. Therefore, choice functions like *uc* and *mc* were introduced. But these do not entirely solve the problem, because often the group of voters use an agenda to reach a decision. An agenda is simply an ordering of the pairs of alternatives from which pairwise comparisons may be made. Now, the choice of a particular agenda is very important, since, using an appropriate agenda, it is possible to end up with Pareto dominated alternatives. For example, consider the tournament T_1. There the alternative c was dominated. But if we use the following agenda, we end up with just this alternative:

alternative b versus a: a is chosen,
alternative a versus d: d is chosen,
alternative d versus c: c is chosen, and is the final choice.

(Note that for any agenda, we always end up in the top cycle.)

To by-pass this problem, Miller (1977), Moulin (1979), Shepsle and Weingast (1982) and others considered so-called sophisticated voting. Without going

into details, it comes down to choosing a binary tree or agenda, such that the outcome ends up in the uncovered set. See Banks (1985) or Moulin (1986) for details. We give the tree for tournaments on X when $|X| = 4$, see figure 2.1, where 1,2,3 and 4 stand for an arbitrary permutation of the elements from X. To illustrate, let T_1 be the circular tournament on $X = \{a,b,c,d\}$ with c as the Pareto dominated alternative, of section 2.3.
If we choose 1 = a, 2 = b, 3 = c and 4 = d, then, as one may easily verify, we end up with the alternative d. If 1 = d, 2 = b, 3 = a and 4 = c, then we obtain b. If we consider all possibilities, we end up with either a, b or d. So, we never end up with the (possibly) Pareto dominated alternative c.

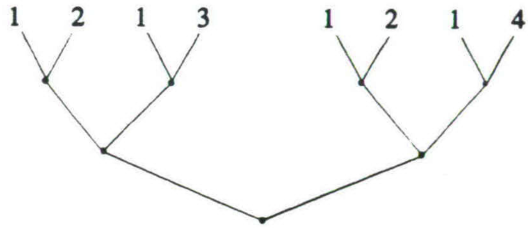


Figure 2.1 The sophisticated agenda for X when $|X| = 4$. (compare Moulin, 1986).

In Banks (1985) it is proved that for any tournament, these outcomes are precisely the elements of $b(T)$. So, using this tree for T_1 as is done above, we see that $b(T_1) = \{a,b,d\}$.

2.5.2 The tournament equilibrium set

In Schwartz (1990) another choice function on $\mathcal{T}$ is proposed. It arises in a cooperative recontracting process. The choice set $S(T,X)$ is interpreted as the set that a majority might make a *final* contract to choose from. Now, take any $x \in X$. Then a *tentative* contract to choose x from X can be upset by a contract to choose something else. What else? Because members act cooperatively, they will cooperatively choose a replacement for x: some majority will make a contract to replace x with an alternative that beats x; a contract that is final in the process of replacing x, though perhaps only tentative in the larger process of choosing from X. Given the interpretation of S, x is replaced with y iff $y \in S(T,\{z \in X: zTx\})$.

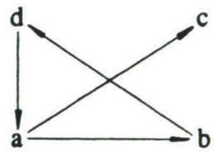
So, suppose a choice function S is defined for all tournaments on Y with $|Y| \leq k$, $k \in \mathbb{N}$, where of course $S(T,\{a\}) = \{a\}$ for all T and all alternatives a. Now, suppose we have a tournament (T,X), with $|X| = k+1$. Based on the fact that S is defined for tournaments (T,Y) with $|Y| \leq k$, we may define a replacement relation D(T,X) on X as follows:

$$aD(T,X)b \text{ iff } a \in S(T,\{y \in X: yTb\}).$$

Note that, because $|\{y \in X: yTb\}| \leq k$, this is well-defined.

Based on the relation D, we may define a choice function *teq*, the tournament equilibrium set, as follows: *teq*(T,X) is the union of the maximal components of the transitive closure of D(T,X).

To illustrate this function, let us again take the circular tournament T_1 on X = {a,b,c,d}.
Since $\{x \in X: xT_1a\} = \{d\}$ and $d \in S(T_1,\{d\})$, we obtain $dD(T_1,X)a$. Moreover, $\{x \in X: xT_1b\} = \{a\}$, hence $aD(T_1,X)b$. Next we have $\{x \in X: xT_1c\} = \{a,b\}$. So, we must determine $S(T_1,\{a,b\})$. Because $\{x \in \{a,b\}: xT_1a\} = \emptyset$ and further $\{x \in \{a,b\}: xT_1b\} = \{a\}$, we obtain $S(T_1,\{a,b\}) = \{a\}$. This means that $aD(T_1,X)c$. Analogously, we have $bD(T_1,X)d$. This results in the following relation $D(T_1,X)$:



It is clear that there is just one maximal component: {a,b,d}. So, $teq(T_1) = \{a,b,d\}$.
The choice set for the tournament T_2 of table 2.1 is equal to $teq(T_2) = \{a,c,g\}$.

We next describe the three characterizing properties for the choice function *teq*.

A subset B of X is called S-retentive in X, where S is an arbitrary choice function, iff no tentative contract to choose an element from B can be upset by a tentative contract to choose an element of X\B. Then the recontracting process, having entered or begun in B, can never depart.

Therefore, Schwartz demands that the set $S(T,X)$ is S-retentive. Moreover, there is no proper subset B of $S(T,X)$ such that B is S-retentive, while $S(T,X)\backslash B$ is not. Finally, if the process began in an S-retentive subset, it would never depart, so the final choice would belong to that subset. Therefore, if B is S-retentive, then $B \cap S(T,X) \neq \emptyset$.

Schwartz proved the following theorem:

THEOREM 2.6 (Schwartz 1990, characterization of *teq*). *A choice function S on* $\mathcal{T}$ *satisfies*

> (i) *for each* $(T,X) \in \mathcal{T}$, $S(T,X)$ *is S-retentive in* (T,X),
>
> (ii) *there does not exist a proper subset B of* $S(T,X)$ *such that B is S-retentive in X, while* $S(T,X)\backslash B$ *is not,*
>
> (iii) *if B is S-retentive in X, then* $B \cap S(T,X) \neq \emptyset$,

iff $S = teq$. ■

2.5.3 The Copeland choice set

Let *cop* be the Copeland choice function. For a tournament T, *cop*(T) consists of those alternatives with highest Copeland score. This score was introduced in chapter 1. If we consider the tournament T_2 of table 2.1, we obtain $cop(T_2) = \{a,b,c\}$. As observed by Miller (1980), for each tournament T, we have $cop(T) \subseteq uc(T)$: if x covers y, then $s_x > s_y$. Unfortunately, as observed by Moulin (1986), the Copeland choice set may be a dominated subset of the uncovered set: there are tournaments T, such that $cop(T) \cap tc{\circ}uc(T) = \emptyset$.

The recursively Copeland choice function: cop*

This choice function is defined in the same manner as was uc^*. For example, $cop^*(T_2) = \{a\}$.

For practical reasons, we postpone characterizations of *cop* to chapter 3, section 3.5.4.

2.5.4 The Slater choice set

Given a tournament T, nearest adjoining orders are linear orders L that, with respect to the tournament T, have a minimal number of differences or upsets. An upset is a pair of alternatives x,y such that yTx while xLy: in L the loosing alternative is ranked ahead of the winning alternative. To give an example, consider the following tournament.

T_3	a	b	c	d	e
a	-	1	1	0	0
b	0	-	1	1	1
c	0	0	-	1	1
d	1	0	0	-	1
e	1	0	0	0	-

There are two nearest adjoining orders:
a » b » c » d » e and
b » c » d » e » a.
With respect to these orders, T_3 has two upsets.

The reason for studying these nearest adjoining orders is the following. Due to disturbances, the tournament contains upsets. The linear orders with minimal number of upsets are the orders representing the rankings that are most likely.

We denote the set of nearest adjoining orders to (T,X) by N(T,X) or N(T). These orders were introduced by Slater (1961).

For the computation of the nearest adjoining orders, we refer to Remage and Thompson (1966) and Phillips (1967, 1969). The computations require exponential computer time. Another computation method is described in Bezembinder (1981).

The nearest adjoining orders for the tournament of table 2.1 are:

a » b » c » d » e » f » g » h,

a » b » c » d » e » g » h » f, a » b » c » e » f » d » g » h,

a » b » c » d » e » h » f » g, a » b » c » f » d » e » g » h.

They all need 5 reversals to be converted into the tournament (T_2,X).

Now, we shall define a choice function, that is based upon these nearest adjoining orders.

Suppose $T \in \mathbf{T}(X)$. Let sl(T) be the set of elements that are best in a nearest adjoining order (sl stands for Slater, 1961).

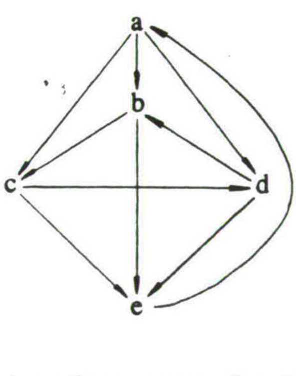
To give an example, $sl(T_2) = \{a\}$.

One may conjecture that for all tournaments T, the intersection of *cop*(T) and *sl*(T) is never empty. Nevertheless, Bermond (1972) provided a tournament on 7 alternatives, for which $cop(T) \cap sl(T) = \emptyset$. So the respective winners may be entirely different.

2.6 THE TOURNAMENT ZERO-SUM GAME

In Laffond et al. (1993) and Fisher and Ryan (1992), a new choice function was introduced, which uses game theoretical arguments. We give an alternative proof for the unicity of the optimal strategy or mixed strategy equilibrium of the tournament zero-sum game and provide a characterization.

Let T_4 be the following tournament.



Consider two candidates who compete for the votes of the electorate through the policy positions they adopt, compare Laffond et al. (1993). The electorate votes and the majority decides for each couple of political issues from {a,b,c,d,e}, which one has to get most attention. These majority votes are represented in the tournament above. Both candidates travel around the country, each day concentrating on one of the political issues. Once they announce this issue, we may use the tournament results to see which one will get most of the credit. We normalize the payoff to 1 for winning, -1 for loosing. If both come up with the same choice, their payoff is 0. This results in the zero-sum game with skew-symmetric matrix A:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & -1 \\ -1 & 0 & 1 & -1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & -1 & -1 & 0 \end{pmatrix}$$

What are the optimal strategies for this tournament game? Later on, we show that the optimal mixed strategy is (1/3,1/9,1/9,1/9,1/3). This means that 1/3 of the time, they have to concentrate on issue a, 1/9 of the time on issue b, etc. Only this strategy has the highest possible guaranteed mean profit, no matter which mixed strategy the other candidate chooses. Note that the alternative e receives more weight than alternatives b, c and d, which, at first sight, is somewhat counterintuitive.

In general, let G be an asymmetric relation on $X = \{x_1,...,x_n\}$. We define $A = (a_{ij})$, a zero-sum nxn-matrix, by:

$a_{ij} = 1$ if $x_i G x_j$, $a_{ij} = -1$ if $x_j G x_i$ else $a_{ij} = 0$, $i,j \in \{1,...,n\}$.

The game corresponding to the skew-symmetric matrix A is symmetric and is called the G-game. In case of tournaments, as in this chapter, we speak of the tournament (zero-sum) game.

We first discuss some elementary topics on zero-sum games. We number the alternatives as 1,2,...,n. A mixed strategy $\underline{q}$ is a vector $\underline{q} \in \mathbb{R}^n$, such that $\underline{q} \geq \underline{0}$ and $\Sigma_{i=1}^{n} q_i = 1$. See the strategy given above. The set of mixed strategies is denoted by K^n.

The number v_1 is called a security level for player 1, if there exists a strategy $\underline{p}$ such that $\underline{p}A \geq v_1\underline{e}$, where $\underline{e}$ is a vector of 1's. By playing $\underline{p}$, the payoff to player 1 at least equals v_1: $\underline{p}A\underline{q}^t \geq v_1\underline{e}\underline{q}^t = v_1$ for each choice of strategy $\underline{q}$.

The number v_2 is called a security level for player 2, if there exists a strategy $\underline{q}$ such that $A\underline{q}^t \leq v_2\underline{e}^t$. By playing this strategy, player 2 loss is less or equal to v_2: $\underline{p}A\underline{q}^t \leq \underline{p}v_2\underline{e}^t = v_2$ for each choice of $\underline{p}$.

Of course, this implies that $v_1 \leq v_2$ for all security levels. Surprisingly enough, there are strategies $\underline{p}^*$ and $\underline{q}^*$, such that the corresponding security levels are equal: $v_1 = v_2$ (von Neumann, 1928). The value $v_1 = v_2$ is called the value of the game, see for example Owen (1982), and is denoted by v or $v(A)$. The strategies $\underline{p}$ such that $\underline{p}A \geq v\underline{e}$, are called optimal strategies for player 1. The set of optimal strategies for player 1 is denoted by $O_1(A)$. For player 2, we analogously obtain $O_2(A)$.

Because the matrix A is skew-symmetric ($A^t = -A$), the G-game is symmetric, and we can prove the lemma below, see Owen (1982).

LEMMA 2.7 *Let $G \in \mathcal{G}(X)$. For the G-game with matrix A, we have that*

(i) $O_1(A) = O_2(A)$

(ii) $v(A) = 0.$

Proof Let $\underline{p}^* \in O_1(A)$, $\underline{q}^* \in O_2(A)$. Then for all $\underline{p},\underline{q} \in K^n$, $n = |X|$:

$$\underline{p}A\underline{q}^{*t} \leq v(A) \leq \underline{p}^*A\underline{q}^t.$$

Transposing these inequalities, we arrive at

$$\underline{q}^*A^t\underline{p}^t \leq v(A) \leq \underline{q}A^t\underline{p}^{*t}.$$

Because $A^t = -A$, we get

$$\underline{q}A\underline{p}^{*t} \leq v(A) \leq \underline{q}^*A\underline{p}^t.$$

This means that $\underline{p}^* \in O_2(A)$ and $\underline{q}^* \in O_1(A)$. Hence $O_1(A) = O_2(A)$.
As for $v(A)$:

$$v(A) = \underline{p}^*A\underline{q}^{*t} = \underline{p}^*(-A^t)\underline{q}^{*t} = -\underline{p}^*A^t\underline{q}^{*t} = -\underline{q}^*A\underline{p}^{*t} = -v(A), \text{ thus } v(A) = 0. \blacksquare$$

Now, since $v(A) = 0$ and $(1/3,1/9,1/9,1/9,1/3)A \geq 0$, we deduce that the strategy $(1/3,1/9,1/9,1/9,1/3)$ is optimal for our starting example T_4, for both candidates.

The optimal strategy for the tournament game corresponding to T_4 appears to be unique. This unicity is no coincidence. Laffond et al. (1993) and Fisher and Ryan (1992) independently proved that for all tournament games there exists a unique mixed strategy equilibrium $(\underline{p},\underline{p})$ or optimal strategy $\underline{p}$. These proofs are quite different. In Laffond et al. the relation $\underline{p}A \geq \underline{0}$ for an optimal strategy $\underline{p}$ is exploited. In the proof given by Fisher and Ryan, the emphasis lies on the fact that the optimal strategy $\underline{p}$ is a null-vector of a submatrix of A.

We present an alternative proof which is also based on the relation $\underline{p}A \geq \underline{0}$.

THEOREM 2.8 *The optimal strategy of a tournament game is unique.*

Proof We first prove: all optimal strategies do have equal support.
Let $\underline{p}$ be an optimal strategy and $Y = \text{supp}(\underline{p})$. Using lemma 2.7, the tournament zero sum game with matrix A is symmetric and has value equal to 0. So we have

$$\underline{p}A \geq \underline{0}. \tag{1a}$$

Note that for arbitrary $\underline{p}$ and $x \in X$, we have that

$$(\underline{p}A)_x = \sum_{y \in Y,\, yTx} \underline{p}_y - \sum_{y \in Y,\, xTy} \underline{p}_y, \text{ so if } x \notin Y \text{ then } \sum_{y \in Y,\, yTx} \underline{p}_y \geq 1/2.$$

We prove:

$$\text{if } x \notin Y \text{ then } \sum_{y \in Y,\, yTx} \underline{p}_y > 1/2.$$

Therefore, we suppose that this sum is equal to 1/2. In deriving a contradiction, we need the following.
Since $a_{ij} \in \mathbb{N}$, $i,j \in \{1,\ldots,n\}$, it is possible to choose optimal strategies $\underline{p}$, such that all values $\underline{p}_y$ are rational. Because $\sum_{y \in Y} \underline{p}_y = 1$, there exists an integer d such that $d\underline{p}_y$ is an integer for all y, and at least one component, say $d\underline{p}_v$, $v \in Y$, is odd. We prove that d is odd.
In general, for a symmetric game with matrix B (having value 0) and optimal strategy $\underline{p}$,

$$(\underline{p}B)_x > 0 \Rightarrow \underline{q}_x = 0, \tag{1b}$$

for all optimal strategies $\underline{q}$.
Hence, because $\underline{p}_v > 0$, we have $(\underline{p}A)_v = 0$ or $(d\underline{p}A)_v = 0$, giving

$$\sum_{y \in Y,\, yTv} d\underline{p}_y - \sum_{y \in Y,\, vTy} d\underline{p}_y = 0.$$

Thus $\sum_{y \in Y,\, yTv} d\underline{p}_y + \sum_{y \in Y,\, vTy} d\underline{p}_y$ is even.

Since this last sum is equal to d - $d\underline{p}_v$, the number d must be odd.

Going back to our assumption, we obtain

$$\sum_{y \in Y,\, yTx} d\underline{p}_y = \tfrac{1}{2}d.$$

But $\frac{1}{2}d \notin \mathbb{N}$, while for all $y \in Y$, $d\underline{p}_y \in \mathbb{N}$, which is a contradiction.

To summarize: if $\underline{p}_x = 0$, then $\sum_{y \in Y,\, yTx} \underline{p}_y > 1/2$ or, putting it differently,

$$\text{if } \underline{p}_x = 0 \text{ then } (\underline{p}A)_x > 0, \tag{1c}$$

which is the converse of (1b).
But now, we are able to prove that all optimal strategies do have equal support, say Y. Let $\underline{p}_1$ and $\underline{p}_2$ be two optimal strategies with support Y_1 and Y_2, such that $Y_1 \neq Y_2$. We derive a contradiction. Since $Y_1 \neq Y_2$, there are optimal strategies $\underline{p}$ and $\underline{q}$, with all components rational, such that their supports Z_1 and Z_2 are different.
If $x \notin Z_1$, then, using (1c), we know that $(\underline{p}A)_x > 0$. Using (1b), this means that for the optimal strategies $\underline{q}$, $\underline{q}_x = 0$. Hence, supp($\underline{q}$) $\subseteq Z_1$. The same way of reasoning shows that supp($\underline{p}$) is contained in supp($\underline{q}$). So, $Z_1 = Z_2$, which contradicts our assumption.

In the second part of the proof, we show that there is just one optimal strategy with support Y.
Consider the linear program for the computation of optimal strategies for B = A+E, E being a matrix of 1's:

$$\text{maximize} \sum_{i=1}^{n} \underline{x}_i$$
$$\text{s.t. } B\underline{x}^t \leq \underline{e}^t.$$

All optimal bases do contain the set Y, because the support of any optimal strategy is Y. Thus, if, in an optimal basis, a basic variable is positive, it remains so in any optimal basis; analogously, if a basic variable is assigned the value 0, it remains 0 for all other optimal bases. So the right-hand-side corresponding to variables in Y is larger than 0, while for a basic variable not in Y, this is equal to 0. Hence, pivots to other optimal bases do not affect the value of any variable. ■

In Laffond et al. (1993) and Fisher and Ryan (1992), the following proposition may be found.

PROPOSITION 2.9 *The number of alternatives in the support of the unique optimal strategy is odd.* ■

We next introduce the choice function *strat* for arbitrary asymmetric relations.

The choice function strat

Let $(G,X) \in \mathbf{G}$. We define

$$strat(G,X) := \bigcup_{\underline{p} \in O(G)} supp(\underline{p}),$$

where $O(G)$ is the set of optimal strategies for the G-game.

In Laffond et al. (1993), this choice function was introduced for tournaments and is called the bipartisan set. As proved above, in case of tournaments the optimal strategy is unique, contrary to the case of arbitrary asymmetric relations, see chapter 4.

The choice set for the tournament T_2 of table 2.1 is *strat*(T_2) = $\{a,c,g\}$.

Note that our example at the beginning of this section shows that the optimal strategy is not necessarily uniform over its support. This indicates that it is possible to further refine the choice set given by *strat*.

We next present a characterization of the choice function *strat* on $\mathbf{T}$.

- **Balanced dominant weights**

  A choice function $S : \mathbb{G}(X) \to 2^X \setminus \{\emptyset\}$ satisfies the balanced dominant weights property, if for all $G \in \mathbb{G}(X)$ the following holds:
  $S(G)$ is support of a distribution of weights $\underline{p} \geq \underline{0}$ summing to 1, such that:

  (I) *if $x \in S(G)$ then* $$\sum_{y \in S(G),\, yTx} \underline{p}_y = \sum_{y \in S(G),\, xTy} \underline{p}_y$$

  (II) *if $x \notin S(G)$ then* $$\sum_{y \in S(G),\, yTx} \underline{p}_y > \sum_{y \in S(G),\, xTy} \underline{p}_y.$$

If a choice function S satisfies the balanced dominant weights property, the coalition $S(G) \subseteq X$ is able to reach a satisfactory division of power. By satisfactory, we mean: it is internally stable, expressed by (I), and it is externally decisive, expressed by (II). So, (I) expresses the balancedness, while (II) stands for the dominance. In chapter 4, we provide a more extensive discussion and show its connection to the von Neumann-Morgenstern stable sets.
There is just one choice function on $\mathbb{T}(X)$ satisfying this condition:

THEOREM 2.10 (Characterization of *strat* on $\mathbb{T}$) *Let S be a choice function on $\mathbb{T}(X)$. Then S = strat iff S satisfies the balanced dominant weights property.*

Proof (only if) Take $Y = \mathrm{supp}(\underline{p})$, where $\underline{p}$ is the unique optimal strategy. Now, for arbitrary x, we have $(\underline{p}A)_x = \sum_{y\,:\,yTx} \underline{p}_y - \sum_{y\,:\,xTy} \underline{p}_y$. Because $\underline{p}$ is the optimal strategy and $\underline{p}_x > 0$ for $x \in Y$, we have $(\underline{p}A)_x = 0$, which proves the balancedness (I). Moreover, using a general theorem from game theory and the fact that the optimal strategy is unique, or using (1.c) of the proof of theorem 2.8, we know that if $x \notin Y$ then $(\underline{p}A)_x > 0$, which proves the dominance (II). (if) Conversely, the balanced dominant weights property implies $(\underline{p}A) \geq \underline{0}$, hence $\underline{p}$ is the unique optimal strategy. Since $S(T) = \mathrm{supp}(\underline{p})$, $S(T) = strat(T)$. ■

From the proof of theorem 2.8, we deduce:

COROLLARY 2.11 *For tournaments T on X, the distribution* $\underline{p}$ *is unique and equals* $\underline{p}^*$. ■

Two characterizing properties for the choice function *mc* were the SSP and the monotonicity. These are also satisfied by *strat*.

PROPOSITION 2.12 (Laffond et al. 1993) *The choice function strat on* $\mathbf{T}$ *satisfies the conditions of SSP and monotonicity.* ■

In Laffond et al. (1991), a comparison is made between *strat* and *mc*. They proved:

PROPOSITION 2.13 *For all* $(T,X) \in \mathbf{T}$, *we have that* $strat(T,X) \subseteq mc(T,X)$. *For some* (T,X), *we have:* $strat(T,X) \subsetneqq mc(T,X)$. ■

From these two propositions, and the characterization of *mc* given in theorem 2.4, we deduce that *strat* does not satisfy the $\hat{\gamma}$ property. We illustrate this using the following tournament, see figure 2.2.

There we have:
$f \in$ *strat*(T,{a,f}), $f \in$ *strat*(T,{a,b,f}), $f \in$ *strat*(T,{a,c,f}),
$f \in$ *strat*(T,{d,f}), $f \in$ *strat*(T,{d,e,f}). But the unique optimal strategy is $\underline{p}$ = (1/5,1/5,1/5,1/5,1/5,0), as may be verified using theorem 2.10. Hence, {a,b,c,d,e} = X\{f} = *strat*(T).

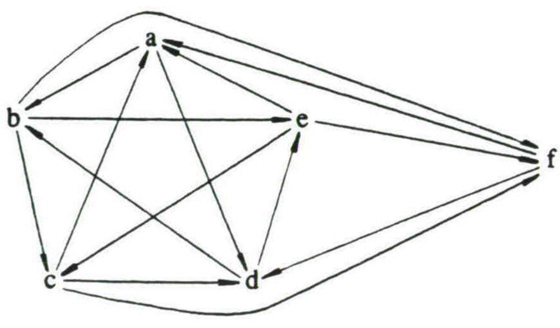


Figure 2.2.

2.7 INDEPENDENCE CONDITIONS

In this section, we consider some other independence conditions. We verify them for a few choice functions. The section is split up in two parts. In the first part, we consider conditions with respect to variable sets of issues. In the second part, conditions with respect to variable relations are introduced.

2.7.1 Variable sets of alternatives

Conditions like SSP, γ (expansion) and Sen's condition, discussed in previous sections, may be categorized as independence conditions. In this section we consider three further independence conditions that are stated in terms of variable sets of alternatives. These are Δ-Independence of irrelevant alternatives, Chernoff's condition and the separability property.

• **Δ-Independence of Irrelevant Alternatives, Δ-IIA**

Before giving the formal definition of the Δ-IIA condition, let us illustrate it using the tournament T_5 in figure 2.3. This tournament may originate from the comparison of 6 distinct alternatives. For example, these alternatives may be dangers that threaten a frigate, such as torpedoes, airplanes, submarines etc. If two such threats occur at the same time, we must choose against which of these defensive actions have to be taken first. All pairwise comparisons of the threats constitute a tournament. It is conceivable that this tournament contains a cycle $a\mathrm{T}b\mathrm{T}c\mathrm{T}d\mathrm{T}a$, as is the case in our example. Aside from the fact that this cycle may be caused by the tension and confusion of the risky situation, there may be structural factors that cause the decisions to be intransitive. For example, the number of characteristics by which the threats can be compared is so large as to necessitate a selection of the criteria. Since this sampling may depend on the threats that are to be compared, there is a possibility of circular behavior, (see also Quandt, 1956).

Now, suppose that we use the choice function cop^* to determine the most threatening danger. Since dangers a and b have the highest Copeland score and aTb, against danger a defensive actions have to be taken first.

Suddenly we are confronted with a new threat, call it ω. Maybe a new ship at the horizon, maybe something that already was at the scene but was not experienced as dangerous, which reveals itself as a potential threat. Assume that in the pairwise comparisons with the other alternatives, it appears to be more dangerous than d,e and f, but less dangerous than a,b and c. At this stage we may apply our choice function cop^* to the new tournament to determine our strategy. According to the Δ-IIA condition, a should be chosen.

T_5

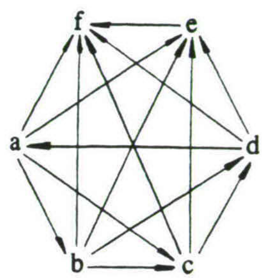


Figure 2.3

Note that in this case, indeed the alternative a is chosen. But, as we will see, in general cop^* does not satisfy the Δ-IIA condition.
Let, if $x \in X$, the set X_x be $X\backslash\{x\}$.

A choice function S on $\mathbf{T}$ *satisfies the Δ-IIA condition, if for all* $(T,X) \in \mathbf{T}$ *and all* $x \in X$:
if aTx *for all* $a \in S(T,X_x)$, *then* $S(T,X) = S(T,X_x)$.

This condition is related to the Δ-IIA condition for ranking rules, described in chapter 3, section 3.5.3.
Unfortunately, neither *cop* nor cop^* satisfy this condition. Consider the following tournament.

T_6	a	x	b	c	d	e	f	g
a	-	1	1	1	1	1	0	0
x	0	-	1	1	1	1	1	1
b	0	0	-	1	1	0	1	0
c	0	0	0	-	0	1	1	1
d	0	0	0	1	-	0	0	1
e	0	0	1	0	1	-	0	1
f	1	0	0	0	1	1	-	0
g	1	0	1	0	0	0	1	-

If we consider the tournament (T_6,X_x), we obtain $cop(T_{6,}X_x) = \{a\}$. We have aTx, but nevertheless $cop(T_6,X) = \{x\}$.

The choice function *uc* also does not satisfy the Δ-IIA condition: consider the tournament (T_7,X_x).

T_7	a	b	c	d	e	f
a	-	1	1	0	0	1
b	0	-	1	0	0	1
c	0	0	-	1	1	1
d	1	1	0	-	1	0
e	1	1	0	0	-	0
f	0	0	0	1	1	-

One may verify that $uc(T_7,X_x) = \{a,c,d\}$. If we insert x between {a,c,d} and {b,e,f} we obtain the tournament T_7 and $uc(T_7) = \{a,c,d,x\}$.

Like the γ condition, the Δ-IIA condition prescribes how to choose from an extended set of alternatives. Note that e.g. SSP is different, in that it prescribes how to choose from a smaller set. In chapter 3 we will use the Δ-IIA condition, actually a weaker form of it, in a characterization for the choice function *cop*. We conclude our discussion of Δ-IIA with its verifications for a few choice functions:

THEOREM 2.14 *The choice functions mc, b and strat, defined on* **T** *are Δ-IIA.*

Proof We start with $S = mc$. Let $M = S(T,X_x)$. This means that (i) $uc(T,M) = M$ and (ii) for all $y \in X_x \backslash M$: $y \notin uc(T,M \cup \{y\})$. Because for all $m \in M$: mTx, it is clear that also $x \notin uc(T,M \cup \{x\})$.

We next prove that *b* is Δ-IIA. Let $a \in b(T,X_x)$, which means that a is top element of a maximal transitive sub tournament of (T,X_x). In T, this subtournament remains maximal, or x can be inserted (not at the top), giving maximality, hence $a \in b(T)$.

Conversely, suppose b is top element of a maximal transitive subtournament of (T,X). We have to consider a few cases.

If x is not in this subtournament, this transitive tournament is also

maximal in (T,X_x) and hence $b \in b(T,X)$.
Now suppose x is in this transitive subtournament that is maximal with respect to inclusion. We assert that x cannot be the top element.
Indeed, if we assume that x is the top element of this maximal subtournament, then all other elements of this subtournament are in $X \backslash b(T)$ because of transitivity. Hence, if we delete x from this subtournament it loses its maximality property, because the top elements of maximal subtournaments in (T,X_x) are in $b(T,X_x)$.
This means that we can extend this transitive subtournament so that it becomes maximal with respect to inclusion. Suppose that there is a transitive extension with an element z at the top. If xTz, then the original subtournament would not have been maximal: insert z between x and the rest. Hence zTx. But now too we may deduce that the original transitive subtournament is not maximal; put z on top of this subtournament.
Now consider a maximal transitive subtournament containing x, with b as its top element, hence $b \in b(T)$. We prove that $b \in b(T,X_x)$. Suppose we delete x from this maximal subtournament, keeping the transitivity property. If it is still maximal in T, we may deduce that $b \in b(T,X_x)$. If not, we can extend it to a maximal transitive subtournament. Is it possible that a part of the extension is at the top, so that b is not top element anymore? Suppose so. Let k be the new top element. It is clear that $k \in b(T,X_x)$. But then k could already have been added to the original subtournament, because kTx, contradicting maximality. Hence b remains the top element, meaning $b \in b(T,X_x)$.
The case S = *strat* is evident if we use theorem 2.10. ∎

Another condition with respect to a variable domain is the Chernoff condition, see e.g. Moulin, 1986:

- **Chernoff's condition**

  A choice function S on **G** *satisfies Chernoff's condition if for all G and all* $X,Y \subseteq U$: *if* $\{Y \subseteq X\}$ *then* $\{S(G,X) \cap Y \subseteq S(G,Y)\}$.

If there is a subset Y such that y is not in the choice set $S(G,Y)$, then there is no set X that contains Y, such that $y \in S(G,X)$ (Nash-IIA). No matter how favorable the relation on X is to y. Positively stated, we may describe this condition as follows: if y is chosen from X, it is chosen from any subset Y of X containing y, which may be seen as a positively stated, reversed γ condition.

It is very strong as is illustrated by the following proposition.

PROPOSITION 2.15 (Moulin 1986). *No choice function S on $\mathcal{T}$ satisfies together Condorcet consistency and Chernoff's condition.*

Proof Take B' = {a,b,c} and T such that its restriction on B' is a cycle aTbTcTa. (Note that we assume that there are at least three elements.) From Chernoff's condition, Condorcet consistency and aTb it follows that $S(T,B') \cap \{a,b\} \subseteq S(T,\{a,b\}) = \{a\}$. Thus $b \notin S(T,B')$. Similar arguments from bTc and cTa show successively that $c \notin S(T,B')$ and $a \notin S(T,B')$. So, $S(T,B') = \emptyset$, which contradicts the assumption of $S(T,B') \in 2^{B'}\backslash\{\emptyset\}$. ■

This Chernoff condition emerges in a study of rationalizable choice functions, see e.g Moulin, 1985. In these studies one has a choice function that picks some outcomes from every subset of a fixed set A of outcomes. When is this function derived from one preference relation on A, hence, when are these choice sets made up of best alternatives within each subset? It appears that the Chernoff condition is a necessary condition for the existence of this relation.

The final condition of this subsection is the separability property.

Separable tournaments (Compare Storcken, 1989)

Suppose that $(T,X) \in \mathcal{T}$. Then the tournament (T,B), $B \subseteq X$, is said to be separable from (T,X), if for all $x \in X\backslash B$:
$(T,B\cup\{x\}) = (T,B) \gg (T,\{x\})$ or $(T,B\cup\{x\}) = (T,\{x\}) \gg (T,B)$.

This means that the alternatives from B behave like a single alternative.

Suppose that we can separate the set of alternatives X into B and X\B, such that for each alternative x from X\B the following is true. Either all alternatives from B are preferred to x, or x is preferred to each alternative in B. In a manner of speaking, the alternatives of B behave like a single alternative. Since in this case, each alternative x from X\B delivers the same local information to all $b \in B$, the local differences between b_1 and $b_2 \in B$ are reduced to their local differences inside B. If we concentrate on local differences, the following condition may be significant.

- **Separability**

  A choice function S on $\mathbb{T}$ *respects separable tournaments, if for all sets* $X \subseteq U$, *all* $(T,X) \in \mathbb{T}(X)$ *and all tournaments* (T,B) *that are separable from* (T,X):

  $$\text{if } S(T,X) \cap B \neq \emptyset \text{ then } S(T,X) \cap B = S(T,B).$$

Note that the separability condition has some resemblance with the Chernoff condition. Compared with the previous proposition, we state:

PROPOSITION 2.16 *There is no choice function S on* $\mathbb{T}$ *satisfying Condorcet transitivity and Condorcet consistency, that respects separable tournaments.*

Proof Consider a circular tournament T_1 on four alternatives $\{a,b,c,d\}$. As one may verify, $(T_1,\{b,c\})$ is separable from T_1. Because of the Condorcet consistency, $S(T_1,\{b,c\}) = \{b\}$. As proved in theorem 2.1, if S satisfies the Condorcet transitivity, then $tc(T_1) \subseteq S(T_1)$. Since $tc(T_1) = \{a,b,c,d\}$, $S(T_1) = \{a,b,c,d\}$. But then, $S(T_1) \cap \{b,c\} \neq S(T_1,\{b,c\})$, which means that S does not respect separable tournaments. ∎

We conclude this subsection with a verification of this condition for a few choice functions in propositions 2.17 and 2.18.

PROPOSITION 2.17 *The choice functions uc, b and mc respect separable tournaments.*

Proof Take $T \in \mathbb{T}(X)$, such that $B \subseteq X$ is a separable subset.
We start with the choice function *uc*. Suppose $A = uc(T)$ and B is separable in T. Assume that $A \cap B \neq \emptyset$. We prove that $uc(T,B) = A \cap B$. Take $b \in B \setminus A$. Hence b is covered in X by an alternative $a \in A$. Suppose $a \in A \setminus B$. Then aTd for all $d \in B$ and a covers all elements from B, because B is separable, including the elements from $A \cap B$. But this gives a contradiction, since

$A \cap B \subseteq A$. Thus b is covered in X by an element from $A \cap B \subseteq B$. This means that $uc(T,B) \subseteq A \cap B$. Conversely, since B is separable in T, $A \cap B$ is uncovered in B. This may be seen by observing that $a \in A \cap B$ is able to reach all alternatives $b \in B$ in no more than two steps, using only elements from B; aTx and xTb, where $x \notin B$ is impossible due to the fact that B is separable.

Suppose $A = b(T)$ and B is separable in T. Assume that $A \cap B \neq \emptyset$. Take $b \in A \cap B$. This element b is top of a maximal transitive sub tournament L of T. Because B is separable, we may write $L = L|B \gg L|(X \setminus B)$, implying that $b \in b(T,B)$. Now take $c \in b(T,B)$. We may extend the maximal transitive sub tournament L' in (T,B) with c as its top element, to a maximal transitive sub tournament in (T,X), by pasting $L|(X \setminus B)$ at the bottom. Hence we obtain $b(T,B) \subseteq A \cap B$.

We now consider the choice function *mc*. Let $A = mc(T)$. We prove that $A \cap B = mc(T,B)$ if $A \cap B \neq \emptyset$.
We first prove that $A \cap B$ is a covering set for (T,B). Therefore, we have to prove that (i) $uc(T,A \cap B) = A \cap B$ and
(ii) for all $b \in B \setminus (A \cap B)$, $b \notin uc(T,(A \cap B) \cup \{b\})$.
First of all, we show that $uc(T,A \cap B) = A \cap B$. Since $mc(T) = A$, we know that $uc(T,A) = A$. Hence, each alternative from A is able to reach all other alternatives from A in no more than two steps, only using elements from A. Now, take an alternative y from $A \cap B$. We prove that y is uncovered in $(T,A \cap B)$. Suppose that xTy, $x,y \in A \cap B$. Since $x,y \in A$, y can reach x in two steps, using an element z from A: yTzTx. But, because B is separable and $x,y \in B$, we have that $z \in B$, meaning that $z \in A \cap B$. Hence y is uncovered in $(T,A \cap B)$. Finally, we have to prove that whenever $b \in B \setminus A$, then $b \notin uc(T,(A \cap B) \cup \{b\})$. Because $A = mc(T)$, we know that $b \notin uc(T,A \cup \{b\})$. This means that starting from b, we cannot reach (in A) all elements from A in maximal two steps. On the other hand, elements from $A \cap B \subseteq B$ are able to reach all elements from A in no more than two steps, because $uc(T,A) = A$. Hence, because B is separable, elements of $A \setminus B$ can be reached from b in maximal two steps. But then, since b cannot reach all elements from A in maximal two steps, b cannot reach all elements of $A \cap B$ in no more than two steps. Thus $b \notin uc(T,(A \cap B) \cup \{b\})$.

To finish the proof, we next show that $A \cap B$ is the minimal covering set of

(T,B).

Consider the set C := A\B $\cup$ *mc*(T,B). Because, as is proved above, we have *mc*(T,B) $\subseteq$ A$\cap$B, we know that C $\subseteq$ A. We show that C is a covering set of (T,X), which means that because of the minimality of A, we have that C = A. But then we obtain *mc*(T,B) = A$\cap$B.

Therefore, we prove:

(i) *uc*(T,C) = C

(ii) for all y $\in$ X\C, y $\notin$ *uc*(T,C$\cup${y}).

Since the proof of (i) is easy if we use the facts that C $\subseteq$ A, *uc*(T,A) = A, B is separable in (T,X) and *uc*(T,*mc*(T,B)) = *mc*(T,B), we only consider (ii).

Let y $\in$ X\C. First, assume that y $\in$ B\C $\subseteq$ B. Since y $\notin$ *uc*(T,*mc*(T,B)$\cup${y}), it is covered in *mc*(T,B) by an element a $\in$ *mc*(T,B). Because B is separable in (T,X), y is covered by a in (T,C$\cup${y}).

Finally, assume that y $\notin$ C$\cup$B. Because y $\notin$ *uc*(T,A$\cup${y}), inside the set A$\cup${y} it is covered by an element a $\in$ A. If a $\in$ (A\B)$\cup$*mc*(T,B), then y is covered by a in C$\cup${y}. Now, suppose that a $\in$ (A$\cap$B)*mc*(T,B). Since y $\notin$ B, we use the separability of B to deduce that y is covered in (T,C) by any element from *mc*(T,B). This proves (ii) and completes the proof for the choice function *mc*. ■

PROPOSITION 2.18 *Let Y = strat(T,X) and* $\underline{p}$ *be the unique optimal strategy. Suppose B is separable in (T,X). Then, if strat(T,X)* $\cap$ *B* $\neq \emptyset$*, we have:*

(a) for all b $\in$ *B:* $\displaystyle\sum_{y\in Y\setminus B\,,\,yTb} \underline{p}_y = \sum_{y\in Y\setminus B\,,\,bTy} \underline{p}_y,$

(b) the unique optimal strategy for (T,B) is proportional to $\underline{p}|B$*, implying that strat(T,B) = strat(T,X)* $\cap$ *B.*

Proof We first prove that (a) and (b) are equivalent. Because B is separable in (T,X), we may reorder the alternatives of X to V$\cup$B$\cup$L, such that V » B and B » L in the tournament (T,X). If we denote the restriction of the zero-sum matrix A to the subsets B, V and L respectively with *B*, *V* and *L*, we obtain

$$A = \begin{pmatrix} V & J & .. \\ -J & B & J \\ .. & -J & L \end{pmatrix}$$

where *J* (-*J*) stands for a block of 1's (-1's), while .. indicates that there are no conditions on the corresponding matrix entries.

If $\underline{p} \in K^n$, then for $Z \subseteq X$, we let $\underline{p}|Z$ be the vector $\underline{p}$ with components outside Z deleted.

Because $\underline{p}$ is an optimal strategy, $\underline{p}A \geq \underline{0}$.

Now, suppose that for all $b \in B$: $\sum_{y \in Y \setminus B, yTb} \underline{p}_y = \sum_{y \in Y \setminus B, bTy} \underline{p}_y$. Then we have for all $b \in B$: $0 \leq (\underline{p}A)_b = \sum_{y \in Y \setminus B, yTb} \underline{p}_y - \sum_{y \in Y \setminus B, bTy} \underline{p}_y + ((\underline{p}|B)\mathit{B})_b = ((\underline{p}|B)\mathit{B})_b$.

Hence $\underline{p}|B$ is proportional to the unique optimal strategy of (T,B).

Conversely, suppose that the optimal strategy of (T,B) is proportional to $\underline{p}|B$. Take $b \in B$ such that $\underline{p}_b > 0$. Then of course, we have $((\underline{p}|B)\mathit{B})_b = 0$. But, since $\underline{p}$ is also optimal in (T,X), we also have $(\underline{p}A)_b = 0$, implying (a) in case $b \in Y \cap B$. Since B is separable, this holds for all $b \in B$.

Now it is sufficient to prove (a). We first show that *strat*$(T,B \cap Y) = B \cap Y$.
We need two results from Laffond et al. (1993) and a composition operation.

We start with the composition. Given is a tournament $(S,Z) \in \mathcal{T}(Z)$, $n = |Z|$ and n tournaments (T_i,Y_i). We denote by $\prod(S;T_1,\ldots,T_n)$ the tournament (S^*,Z^*) obtained from the composition of S and T_i, $i = 1,\ldots,n$. The set Z^* is the disjoint union of the sets Y_i, and S^* is defined by:

aS^*b iff (1) a and b belong to the same Y_i and aT_ib, or
(2) a belongs to Y_i and b belongs to Y_j, $i \neq j$ and aSb.

The tournaments T_i are called components and S is called the summary of the composed tournament.

The two results from Laffond et al. (1993) are

Corollary 2. Let $\underline{p}$ be the unique optimal strategy, with support Y. Then (T,Y) is the summary of a tournament (Z,V), in which all elements have equal score. Moreover, for each $y \in Y$, $\underline{p}_y$ equals the proportion of the component y in (Z,V)

Proposition 2. Let (T,X) be the summary of a tournament (Z,V) in which all n elements have equal score. Let d(x) be the order of the component x. Then *strat*(T,X) = X and the unique optimal strategy is equal to $\underline{p}$, where $\underline{p}_x = d(x)/n$.

Using corollary 2, we know that, since $Y = \mathit{strat}(T,X)$, (T,Y) is the summary of a tournament (T^*,Y^*) in which all alternatives have the same score. Moreover, for each $y \in Y$, $\underline{p}_y$ represents the proportion of the component y in (T^*,Y^*).

Because B is separable in (T,X), all elements of $(T^*,(B\cap Y)^*)$ have the same score. Hence, $(T,B\cap Y)$ is the summary of a tournament in which all alternatives do have the same score. From proposition 2, we may deduce that $\mathit{strat}(T,B\cap Y) = B\cap Y$, and that the optimal strategy is proportional to $\underline{p}|(B\cap Y)$. Let B' be the restriction of A to $B\cap Y$.

Because of this proportionality, we know that $(\underline{p}|B\cap Y)B' = \underline{0}$. So, we deduce that, since $(\underline{p}A)_b = 0$ for $b \in B\cap Y$,

$$\text{for all } b \in B\cap Y: \quad \sum_{y\in Y\setminus B,\, yTb} \underline{p}_y = \sum_{y\in Y\setminus B,\, bTy} \underline{p}_y,$$

which implies (a), because B is separable. ■

2.7.2 Variable relations

In this subsection, we consider two independence properties that are stated in terms of variable relations. Both conditions are used in chapter 4.

- **Global monotonicity**

A choice function S on $\mathbb{G}(X)$ is globally monotone, if for all asymmetric relation $(G,X) \in \mathbb{G}(X)$ and all $x,y \in X$,

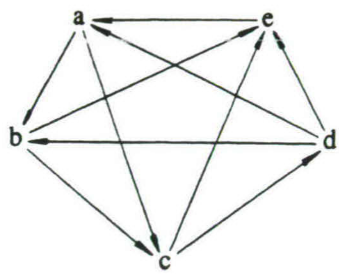
if $x \in S(G,X)$, $y \notin S(G,X)$, while $(x,y) \notin G$,

then $S(G,X) = S(G',X)$, where $G' = [G\setminus\{(y,x)\}] \cup \{(x,y)\}$.

Suppose that the choice function S on $\mathbb{G}(X)$ is globally monotone. Let $(G,X) \in \mathbb{G}(X)$. Then, adjusting preferences in the 'right direction', inspired by the ranking $S(G,X) \gg X\setminus S(G,X)$, does not change the choice set. If, for example $x \in S(G,X)$, $y \notin S(G,X)$ and yGx, then the choice set does not change if we reverse the relation between x and y to xG'y.

The distinction to the monotonicity property used in the characterization of the choice function *mc*, is the following. In the formulation of the global monotonicity, we not only state a condition for x, but also for all members of the choice set. Moreover, y has to be an element of $X \setminus S(G,X)$.

It is a strong condition. The choice functions *uc*, *b* and *cop* do not fulfill this condition. We show this for *uc* and *b*.
Let T_8 be the following tournament on $X = \{a,b,c,d,e\}$.



As one may verify, the alternative e is covered by d. Moreover, $uc(T_8) = b(T_8) = \{a,b,c,d\}$. Now, let $T'_8 = T_8 \setminus \{(e,a)\} \cup \{(a,e)\}$. Then a covers b in T'_8, hence $uc(T_8) \neq uc(T'_8)$. The same holds for the choice function *b*. For these choice functions we have: what is good for a member of a coalition, in our case alternative a in the coalition $\{a,b,c,d\}$, is not necessarily profitable for the total coalition. In our case, it is not good for b.

PROPOSITION 2.19 *The choice functions tc, mc and strat on* $\mathbf{T}$, *are globally monotone.*

Proof We only consider the case $S = mc$. The other two cases are easy. Take $T \in \mathbf{T}(X)$ and let $A = mc(T)$. Assume that $x \in A$, $y \in X \setminus A$ and yTx. Let $T' = [T \setminus \{(y,x)\}] \cup \{(x,y)\}$. We prove that $mc(T') = A$.
It is clear that $uc(T',A) = A$, since $(T',A) = (T,A)$. Moreover, $y \notin uc(T,A \cup \{y\})$, because $A = mc(T)$. Again, it is clear that $y \notin uc(T',A \cup \{y\})$. Thus A is a covering set of T'. We prove that it is the minimal covering set.

Because $A = mc(T)$ is a covering set for T', we have:

$$mc(T') \subseteq mc(T). \qquad \text{(I)}$$

In Dutta (1988), it is proved that *mc* satisfies SSP. So, because of (I), we obtain

$$mc(T') = mc(T',mc(T)).$$

Because the differences between T and T' are outside $mc(\mathrm{T})$, we have that

$$mc(\mathrm{T}', mc(\mathrm{T})) = mc(\mathrm{T}, mc(\mathrm{T})). \qquad \text{(IIa)}$$

Moreover, again using SSP, we obtain

$$mc(\mathrm{T}, mc(\mathrm{T})] = mc(\mathrm{T}). \qquad \text{(IIb)}$$

So, $mc(\mathrm{T}) = mc(\mathrm{T}')$. ■

• **Tail stability**

The last condition of this section is the tail stability. To explain it, we consider the ranking $S(\mathrm{G,X}) \gg \mathrm{X} \backslash S(\mathrm{G,X})$ of an asymmetric relation (G,X). If we change comparisons between alternatives not in $S(\mathrm{G,X})$, the tail stability of S means that there is no need to adjust the choice set; nothing will change. So, the choice set is stable with respect to changes in the tail of the ranking mentioned above.

We also may explain it using the relation matrix (see chapter 1) in figure 2.4. If changes are limited to the shaded area, the choice set will not change.

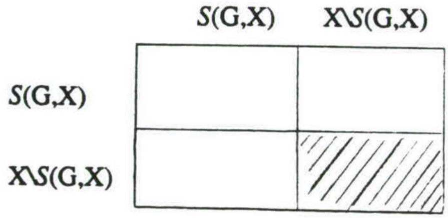


Figure 2.4 Changes in the shaded area do not have any impact on the choice set $S(\mathrm{G,X})$, S being tail stable.

We finally present the formal definition:

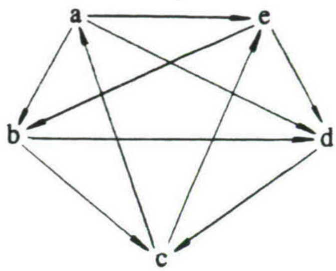
A choice function S on $\mathcal{G}(X)$ is tail stable, if for all asymmetric relations (G,X) and (G',X) on X,

$$\text{if } G \cap \left(X^2 - (X - S(G,X))^2\right) = G' \cap \left(X^2 - (X - S(G,X))^2\right) \text{ then}$$
$$S(G,X) = S(G',X).$$

We have the following proposition. The proof is easy and is left to the reader.

PROPOSITION 2.20 *The choice functions tc, mc and strat are tail stable.* ∎

Note that the choice functions b and uc do not satisfy the condition of tail stability. To illustrate this, let T_9 be equal to:



Then $uc(T_9) = b(T_9) = \{a,b,c\}$. If we reverse the comparison between e and d to dT'_9a, we obtain $uc(T'_9) = b(T'_9) = \{a,b,c,d\}$.

We conclude this section with a connection between some conditions stated in terms of a variable relation, and SSP, a condition stated in terms of a variable set of issues.

PROPOSITION 2.21 *Let a choice function S on $\mathcal{G}$ be globally monotone, tail stable and concatenation consistent. Then S satisfies SSP.*

Proof Suppose that $S(G,B') \subseteq B \subseteq B'$. We have to prove that $S(G,B) = S(G,B')$. Because S is globally monotone and is tail stable, we may change the relation (G,B') to (G',B') such that $(G',B') = (G,B) \gg (G,B'\setminus B)$, while $S(G',B') = S(G,B')$. Since S is concatenation consistent, we obtain $S(G',B') = S(G,B)$. ∎

As a corollary, we present

COROLLARY 2.22 (Characterization of *mc*) *The choice function mc is the smallest choice function satisfying (simultaneously) global monotonicity, tail stability, concatenation consistency, neutrality, monotonicity and* $\hat{\gamma}$.

Proof Use proposition 2.21 and theorem 2.4. ∎

2.8 SET THEORETICAL COMPARISON, TABLE OF PROPERTIES

In this final section, we consider set-theoretical comparisons between various choice functions and give a table of properties.

Set-theoretical comparisons of the various sets *S*(T) may be found in for example Banks (1985), Banks et al. (1991), Dutta (1988), Laffond et al. (1992), Moulin (1986) and Schwartz (1990). We present the following table from Laffond et al. (1992).

	tc	*uc*	*tc∘uc*	uc^*	*cop*	*sl*	*b*	b^*	*mc*	*teq*
tc										
uc	$\subseteq$									
tc∘uc	$\subseteq$	$\subseteq$								
uc^*	$\subseteq$	$\subseteq$	$\subseteq$							
cop	$\subseteq$	$\subseteq$	$\varnothing$	$\varnothing$						
sl	$\subseteq$	$\subseteq$	$\subseteq$	$\varnothing$	$\varnothing$					
b	$\subseteq$	$\subseteq$	$\subseteq$	$\cap$	$\varnothing$	$\varnothing$				
b^*	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\varnothing$	$\varnothing$	$\subseteq$			
mc	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\varnothing$	$\varnothing$	$\cap$	•		
teq	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\varnothing$	$\varnothing$	$\subseteq$	$\subseteq$	•	
strat	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\varnothing$	$\varnothing$	•	•	$\subseteq$	•

Explanation:
- (i) S $\subseteq$ S' iff for all T $\in$ $\mathbb{T}$(X): S(T) $\subseteq$ S'(T).
- (ii) S $\cap$ S' iff not S $\subseteq$ S' and not S' $\subseteq$ S, and for all T $\in$ $\mathbb{T}$(X): S(T) $\cap$ S'(T) $\neq$ $\varnothing$.
- (iii) S $\varnothing$ S' iff S(T) $\cap$ S'(T) = $\varnothing$ for some T.
- (iv) S • S' iff the relation between S and S' is yet unknown.

Table 2.2 From Laffond et al. (1992).

Of course, using this table, we may construct new choice functions. For example, since for all tournaments T, $b(T) \cap mc(T)$ is nonempty, we may define $S_{b \cap mc}(T) = b(T) \cap mc(T)$. This choice set combines the attractive features of both elements from $b(T)$ and $mc(T)$.

To give an interesting relation between $sl(T)$ and $uc(T)$, we mention the following theorem.

THEOREM 2.23 (Banks et al. 1991). *Suppose $(T,X) \in \mathcal{T}(X)$. Let (C,X) be the cover relation. If (L,X) is a nearest adjoining ordering of (T,X), then $C \subseteq L$.* ■

Now it is easy to prove that $sl(T,X) \subseteq uc(T,X)$ for all $(T,X) \in \mathcal{T}$.

We conclude this chapter with a table of properties.

	Δ-IIA	separable	global monotone	tail stable	SSP	γ
uc	□	■	□	□	□	■
b	■	■	□	□	□	□
mc	■	■	■	■	■	□
strat	■	■	■	■	■	□

□: not satisfied

■: satisfied

Table 2.3 Table of properties.

CHAPTER 3

RANKING AND RATING TOURNAMENTS

3.1 INTRODUCTION

Given is a finite set $U = \{u_1,...,u_k\}$ of alternatives. In this chapter, we consider the ranking of a finite collection $X = \{x_1,...,x_n\} \subseteq U$ of distinct alternatives, by way of pairwise comparison. We assume that the alternatives are value-distinguishable, so that in each comparison of two distinct alternatives, a strict preference for one of them is revealed. In this chapter we concentrate on rules that rank tournaments arising from the direct confrontation between various alternatives or players, for example players in a chess tournament. Most conditions and assumptions in this chapter, are significant in this context.

If a tournament is free of cycles, a natural ranking does exist: there is a best alternative, a second best etc. In the nontransitive case, there are more possibilities than just one, due to the existence of cycles. All rules that will be introduced and discussed deal with these cycles differently.

Ranking rules

A ranking rule f on $\mathbb{T}(X)$ is a mapping $f : \mathbb{T}(X) \to \mathbb{W}(X)$. A ranking rule f on $\mathbb{T}$ is a family of ranking rules, one for each $\emptyset \neq X \subseteq U$.

3.2 ELEMENTARY CONDITIONS

Throughout this chapter, we assume that the ranking rules satisfy some elementary conditions.

We start with the commutation with concatenation. Later on, we will introduce two other commutation properties for ranking rules, commutation with respect to permutations and conversion. These three properties were introduced in Delver, Monsuur, Storcken (1991).

• **Commutation with concatenation**

A ranking rule f on $\mathbb{T}$ commutes with concatenation if for all Y and Z such that $Y \cap Z = \emptyset$, for all $(T,Y) \in \mathbb{T}(Y)$ and all $(T,Z) \in \mathbb{T}(Z)$:

$$f[(T,Y) \gg (T,Z)] = f(T,Y) \gg f(T,Z).$$

This condition assures that the preferences $yTz \in Y \times Z$, contained in the concatenation $(T,Y) \gg (T,Z)$ are preserved. Furthermore, the preference of $f[(T,Y) \gg (T,Z)]$ in Y (or Z) only depends on (T,Y) (or (T,Z)). Since $(T,Y) \gg (T,Z)$ implies that (T,Y) and (T,Z) may be ranked independently, this is a reasonable condition. Compare this with the concatenation consistency for choice function, introduced in chapter 2.

Note that since every tournament (T,X) may uniquely be written as $(T,X) = (T,X_1) \gg ... \gg (T,X_n)$, where the sets X_i, $i = 1,...,n$, are the irreducible parts of (T,X), we have $f(T,X) = f(T,X_1) \gg ... \gg f(T,X_n)$ whenever f commutes with concatenation.

• **Commutation with permutation**

Let σ be a permutation of X. For all relations R on X, σR is a relation on X defined by: $(\sigma(x),\sigma(y)) \in \sigma R$ iff $(x,y) \in R$.

A ranking rule f on $\mathbb{T}$ commutes with permutations if for all $X \subseteq U$ and for all permutations σ of X and all tournaments $T \in \mathbb{T}(X)$

$$f(\sigma T) = \sigma f(T).$$

We impose this condition upon f to exclude the possibility that f makes a distinction between different ways of naming the alternatives. Now, let $T \in \mathbb{T}(X)$ and suppose that two alternatives a and b *play the same role in T*, which means that there is a permutation σ such that $\sigma T = T$ and $\sigma(a) = b$. In proposition 3.1 we prove that for any ranking rule f commuting with permutation, the two alternatives are indifferent: $a \approx b : f(T)$. To give an example, suppose T' is a simple 3-cycle on $X = \{a,b,c\}$. So we have $T' = \{(a,b),(b,c),(c,a)\}$. If $\sigma(a) = b$, $\sigma(b) = c$ and $\sigma(c) = a$ then $\sigma T' = \{(b,c),(c,a),(a,b)\} = T'$. Hence, using proposition 3.1, we see that $a \approx b : f(T')$. Since in this example all alternatives play the same role, any f that commutes with permutation maps T' onto a single indifference class.

PROPOSITION 3.1 *Let $a, b \in X$. If a ranking rule f on $\mathbf{T}$ commutes with permutations, and alternative a plays the role of alternative b in $T \in \mathbf{T}(X)$, then $a \approx b : f(T,X)$.*

Proof Let $T \in \mathbf{T}(X)$ and $a,b \in X$. Take σ such that $\sigma T = T$ and $\sigma(a) = b$. Suppose that $(a,b) \in f(T)$. Then $(\sigma(a),\sigma(b)) \in \sigma f(T) = f(\sigma T) = f(T)$. In general $(\sigma^i(a),\sigma^i(b)) \in \sigma^i f(T) = f(T)$. Because σ is a bijection and $|X|$ is finite, there is a k such that $\sigma^k(b) = a$. So we have the chain (a,b), $(\sigma(a),\sigma(b))$, ..., $(\sigma^k(a),a) \in f(T)$, in which $\sigma^i(a) = \sigma^{i-1}(b)$. The transitivity of $f(T)$ gives: $(b,a) \in f(T)$. Similarly $(b,a) \in f(T)$ gives $(a,b) \in f(T)$, so that $a \approx b : f(T)$. ■

From proposition 3.1, we deduce that if f commutes with permutations, $f(\mathbf{T}(X))$ is not a subset of $\mathbf{L}(X)$. Since commuting with permutations is very natural, this explains why we have to accept that, in general, $f(T,X)$ is a weak order.

Note that for all ranking rules $f : \mathbf{T}(X) \to \mathbf{W}(X)$, some loss of information must be accepted, in the sense that a given image does not always determine a unique original: in general, there are more tournaments that weak orders; f is not injective.

3.3 INDUCED RANKING RULES

In this section, we introduce the class of ranking rules that are induced by a choice function. Moreover, we present a characterization for this class.

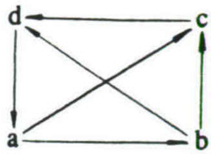
Given a choice function S on $\mathbf{G}$, we may define the induced ranking rule f_S on $\mathbf{G}$:

Let S be a choice function defined on $\mathbf{G}$. Then the induced ranking rule f_S on $\mathbf{G}$ is defined as follows. For all $(G,X) \in \mathbf{G}$,

$$\text{if } X \backslash S(G,X) = \varnothing \text{ then } f_S(G,X) = S(G,X), \text{ else}$$
$$f_S(G,X) = S(G,X) \gg f_S(G,X \backslash S(G,X)).$$

Not every ranking rule is induced by a choice function. An example is provided by the rule that orders the alternatives to their Copeland score (f_{out} to be discussed in section 3.5). To prove this, suppose $f_{out} = f_S$ where S is a choice function. Consider the tournament T_1.

T_1



Since $f_{out}(T_1) = \{a,b\} \gg \{c,d\}$, $S(T_1)$ consists of the two alternatives $\{a,b\}$ with score equal to 2. Now, if $S(T_1,\{c,d\}) = \{c\}$, then $S(T_1,\{c,d\}) \neq \{c,d\} = f_{out}(T_1)|\{c,d\}$, meaning that f_{out} is not induced by S. If $S(T_1,\{c,d\}) = \{c,d\}$, then $\{c,d\}\setminus S(T_1,\{c,d\}) = \emptyset$, so we must have $f_{out}(T_1,\{c,d\}) = S(T_1,\{c,d\})$, which is not the case, since $f_{out}(T_1,\{c,d\}) = \{c\}$.

We give a characterization of ranking rules that are induced by a choice function. For that purpose, we need the notion of a right-interval. Suppose $W = X_1 \gg X_2 \gg \ldots \gg X_n$ is a weak order. We may define a right-interval of W as the interval $Y = X_i \cup \ldots \cup X_n$, $i \in \{1,\ldots,n\}$.

If a ranking rule f is induced by a choice function S, it satisfies the following independence condition.

- **Right-interval consistent**

  A ranking rule f on $\mathbb{G}$ is right-interval consistent, if for all relations $(G,X) \in \mathbb{G}(X)$ and all right-intervals Y of $f(G,X)$, we have: $f(G,X)|Y = f(G,Y)$.

This means that the ranking of the alternatives in the tail of the ranking produced by f, is independent of the presence of alternatives in the top of the ranking.

Conversely, it completely characterizes those ranking rules:

THEOREM 3.2 *A ranking rule f on $\mathbb{G}$ is induced by a choice function, iff f is right-interval consistent. In that case $f = f_S$, where S = Best(f).*

Proof (only if) This is evident from the definition of the S-induced ranking rule. (if) Let $G \in \mathbb{G}(X)$. Define S = Best(f): $S(G,X)$ = Best($f(G,X)$). Now, let $f(G) = X_1 \gg \ldots \gg X_n$ and $Y = X \backslash X_1$. Then $f(G) = S(G) \gg f(G)|Y$. But, because $f(G)|Y = f(G,Y)$, this is equal to $f(G) = S(G) \gg f(G,Y) = S(G) \gg f(G,X \backslash S(G))$. ∎

We refer to sections 3.6, 3.7 and chapter 4, section 4.5.2 for more results concerning induced ranking rules.

3.4 THE TRANSITIVE CLOSURE

We introduce the ranking rule f_{trans}, which assigns to a tournament its transitive closure. We present two characterizations.

Let (T,X) be a tournament with irreducible parts (T,X_1), ..., (T,X_n), such that $(T,X) = (T,X_1) \gg \ldots \gg (T,X_n)$.
We define $f_{trans}(T,X) := X_1 \gg X_2 \gg \ldots \gg X_n$. It is called the *transitive closure* of (T,X), see Roubens and Vincke (1985). Also see chapter 1.

A serious drawback of the ranking rule f_{trans} is the fact that it produces large indifference classes. This may be seen by taking a linear order and reversely connecting top and bottom element. To f_{trans} then, all elements become indifferent. To give yet another example, consider the tournament T of table 3.1, section 3.5.1. There we have the cycle aTb, bTc, cTd, dTe, eTf, fTg, gTh and hTa. Thus $f_{trans}(T)$ = {a,b,c,d,e,f,g,h}. From these examples, it is clear that the ranking rule f_{trans} has little discriminating power and will often have to be refined.

On the other hand, it may be seen as one of the most natural ranking methods, because it is the only one that satisfies the conditions mentioned in theorem 3.3. In preparation of this theorem, we introduce:

• **Free of upsets**

A ranking rule f is free of upsets if for all X, for all tournaments (T,X) and all $x,y \in X$: if xTy then $x \geq y : f(T,X)$.

This condition may also be formulated as $(T,X) \subseteq f(T,X)$ for all $(T,X) \in \mathbf{T}$.

THEOREM 3.3 (Characterization of f_{trans}). *Let f be a ranking rule on* $\mathbf{T}$. *Then $f = f_{trans}$ iff f commutes with concatenation and is free of upsets.*

Proof (only if) Evident. (if) Let T be an irreducible tournament on X. Since f commutes with concatenation, it is sufficient to prove that $f(T) = X$. Now, because T is irreducible, there exists an Hamilton path $x_1,x_2,\ldots,x_n,x_1$ in T. Because f is free of upsets, it follows that $x_1 \geq x_2 \geq \ldots \geq x_n \geq x_1 : f(T)$. Hence $f(T) = X$. ■

Compare this theorem with theorem 2.1 of chapter 2.
If we do not consider the ranking produced by f_{trans} as a serious candidate for the final ranking of the alternatives from X, we have to accept the phenomenon of upsets. We have to allow that there are alternatives a and b such that aTb, but nevertheless $b \gg a : f(T)$. This means that there will be friction between direct comparisons at the tournament level and final rankings, which in fact may be seen as a balancing of information from the tournament, especially a balancing of pairwise comparisons along cycles. Summarizing, we have to allow that two levels of analysis, the local and the global view, may give rise to different conclusions.
Of course, this balancing of binary comparisons has to be reasonable. To exclude arbitrariness, we look at desirable properties and characterizations of ranking rules.

But first, we give another characterization of f_{trans}, which was presented in Delver, Monsuur and Storcken (1991). There the notion of *independence* of interval changes was introduced:

• **Independence of interval changes**

Suppose we have a ranking of the alternatives in a tournament (T,X): $f(T,X) = X_1 \gg X_2 \gg \ldots \gg X_i \gg \ldots \gg X_j \gg \ldots \gg X_n$. We may define an interval as the union of successive indifference classes. In the example we choose the interval $Y = X_i \cup ... \cup X_j$. Next, (T,Y) is modified to (T',Y). Relations between alternatives outside Y remain as they were, also the connection between Y and the remaining alternatives is kept the same. Thus we obtain a new tournament (T',X). A ranking rule f is said to be independent of interval changes if the changes of ranking are limited to Y, and occur according to f(T',Y). More precise, f is independent of interval changes if we obtain $f(T',X) = X_1 \gg .. \gg X_{i-1} \gg f(T',Y) \gg X_{j+1} \gg .. \gg X_n$. As may be easily verified, f_{trans} is independent of interval changes.

To be more precise, let $Y \subseteq X$, $(T,X) \in \mathbb{T}(X)$ and $(T',Y) \in \mathbb{T}(Y)$. Then

$$\mathrm{Sub}((T,X),(T',Y)) := \{(x,y) \in X{\times}X : (x,y) \in Y{\times}Y \cap T' \text{ or } (x,y) \notin Y{\times}Y \text{ and } (x,y) \in T\} \in \mathbb{T}(X).$$

A ranking rule h on $\mathbb{T}$ *is said to be independent of interval changes, if for all* $X \subseteq U$ *and all* $(T,X) \in \mathbb{T}(X)$*, all intervals Y of h(T,X) and all* $(T',Y) \in \mathbb{T}(Y)$: $h(\mathrm{Sub}((T,X),(T',Y))) = \mathrm{Sub}(h(T,X),h(T',Y))$.

Note that this condition implies the right-interval consistency of section 3.3.

THEOREM 3.4 (Characterization of f_{trans}) *Let h be a ranking rule on* $\mathbb{T}$.
Then $h = f_{trans}$, *if and only if*
h is independent of interval changes and
h commutes with concatenation.

Proof The "only if" part being straightforward, we only prove the "if" part. Suppose that h satisfies these two conditions. Let $T \in \mathbb{T}(X)$ and let $X_1 \gg X_2 \gg .. \gg X_n = h(T)$. By the commutation with the concatenation we know that (T,X_i) is irreducible, hence has a cycle containing all its alternatives. It is sufficient to prove that there is no cycle involving several indifference classes of $h(T)$. Suppose $a \in X_1$, $b \in X_j$, $j > 1$ and bTa. We deduce a contradiction and are done. Let $(L_1,X_1) \in \mathbb{L}(X_1)$ be such that a is its top element. Take $(L_2,X_a) \in \mathbb{L}(X_a)$ ($= \mathbb{L}(X-\{a\})$) such that b is its top element. Note that b is the top element of

$$\mathrm{Sub}(\mathrm{Sub}((T,X),(L_1,X_1)),(L_2,X_a)). \tag{1}$$

X_1 is an interval of $h(T,X)$. Hence, by the independence of interval changes it follows that .

$$h(\mathrm{Sub}((T,X),(L_1,X_1))) = \mathrm{Sub}(h(T,X),h(L_1,X_1)) =$$
$$\{a\} \gg (L_1,X_1\text{-}\{a\}) \gg X_2 \gg .. \gg X_n.$$

So X_a is an interval of $h(\mathrm{Sub}(T,X),(L_1,X_1))$. Using the independence of interval changes, we obtain

$$h(\mathrm{Sub}(\mathrm{Sub}((T,X),(L_1,X_1)),(L_2,X_a))) =$$

$$\mathrm{Sub}(\mathrm{Sub}(h(T,X),h(L_1,X_1)),h(L_2,X_a)) = \{a\} \gg \{b\} \gg (L_2,X_a\text{-}\{b\}). \quad (2)$$

From (1) and (2) it follows that h does not commute with the concatenation. This contradicts our assumptions. ∎

It is easy to verify that the conditions of theorem 3.4 are independent.

3.5 COPELAND SCORES

In this section, we discuss the Copeland scoring rules. In the first subsection, we introduce them and show how to compute the rankings. In section 3.5.2, we characterize the ranking rule f_{out}. Section 3.5.3 is devoted to a particular independence condition, which will be used in section 3.5.4. Rankings and choice sets that only depend on the score vector, are characterized in section 3.5.4. In the same section, we present characterizations of the Copeland choice function and a related one. In section 3.5.5, we discuss some results that may be found in the literature.

3.5.1 The Copeland scoring rules

The Copeland-score rule: f_{out}

The rule f_{out} is defined as follows: Let $T \in \mathbb{T}(X)$. Then

(1) $x \gg y : f_{out}(T) \Leftrightarrow s_x(T) > s_y(T)$ and
(2) $x \approx y : f_{out}(T) \Leftrightarrow s_x(T) = s_y(T)$.

The algorithm f_{out} is very popular. It is easy to compute and easy to explain. It is suited for all kinds of outcome tournaments, like tennis and football tournaments. Since f_{out} only uses the scores of the alternatives, which is equal to the number of defeated opponents, it stimulates the fighting spirit.

The ranking rule induced by the Copeland choice set: f_{cop}

Let $cop(T)$ be the Copeland choice set, which consists of alternatives that have the largest outscore or Copeland-score. If these champions withdraw from the tournament, for example because they are promoted to a higher division in the football league, the ranking produced by f_{out} may not be satisfactory for some of the players that stay behind. It is possible that one of the new first ranked players is different from the original first runner up, see for example T_1. In that tournament the alternatives c and d become winners, after removal of the alternatives a and b. This means that

$f_{out}(T,X) \mid [X \setminus cop(T)] \neq f_{out}(T,X \setminus cop(T,X))$ in general. I.e. f_{out} is not induced by cop.

To by-pass this problem, we consider the rule f_{cop}, induced by cop:
This ranking rule determines the champions $cop(T,X)$, and then only uses the information contained in $(T,X \setminus cop(T,X))$ to further rank the remaining alternatives. Now we have: $f_{cop}(T,X) \mid [X \setminus cop(T,X)] = f_{cop}(T,X \setminus cop(T,X))$.

Recursive application

For each ranking rule f on $\mathbf{T}$, we may introduce a sequence of maps f^1, f^2, ..., on $\mathbf{T}$ as follows:

$f^1 := f$ and
f^k for all $k \geq 2$ is defined as follows:

if $f^{k-1}(T,X) = X_1 \gg X_2 \gg ... \gg X_t$ then

$f^k(T,X) := f(T,X_1) \gg f(T,X_2) \gg .. \gg f(T,X_t)$.

Hence, in a manner of speaking, for all $k \in \mathbb{N}$: $f^k = f \circ f^{k-1}$.
Since we assume that $U = \{u_1,...,u_n\}$ is finite, the number of tournaments on X is finite. Hence there exists a number k^* such that $f^{k^*} = f^k$ for all $k \geq k^*$. We define $f^* = f^{k^*}$.
The rule f^* is called the *recursive application* of f. Note that in further

refining an indifference class I produced by f, it only uses the information contained in the tournament (T,I).

f^*_{out}

In order to obtain a further ranking of the indifference classes of an image produced by f_{out}, it may recursively be applied to its own indifference classes. For example, if more than one player has the highest score, f^*_{out} sometimes produces a unique winner.

To give an example, we look at T_1.
There we have $s_a = s_b = 2$ and $s_c = s_d = 1$.
Since $f_{out}(T_1,\{a,b\}) = \{a\} » \{b\}$ and $f_{out}(T_1,\{c,d\}) = \{c\} » \{d\}$, we obtain

$f^*_{out}(T_1) = \{a\} » \{b\} » \{c\} » \{d\}$.

The following table contains rankings of T_2, produced by the various ranking rules introduced so far, where f^*_{cop} is the recursive application of f_{cop}.

T_2	a	b	c	d	e	f	g	h
a	-	1	1	1	1	1	0	0
b	0	-	1	1	1	1	0	1
c	0	0	-	1	1	1	1	1
d	0	0	0	-	1	0	1	1
e	0	0	0	0	-	1	1	1
f	0	0	0	1	0	-	1	0
g	1	1	0	0	0	0	-	1
h	1	0	0	0	0	1	0	-

f_{trans}	f_{out}	f^*_{out}	f_{cop}	f^*_{cop}
{a,b,c,	{a,b,c}	{a}	{a,b,c}	{a}
d,e,f,	{d,e,g}	{b}	{d,e}	{b}
g,h}	{f,h}	{c}	{f,g,h}	{c}
		{d}		{d}
		{e}		{e}
		{g}		{f,g,h}
		{h}		
		{f}		

Table 3.1. Various rankings of T_2.

3.5.2 A characterizations for f_{out}

This section is based on Delver, Monsuur, Storcken (1991).

In preparation of a characterization of f_{out}, we need a few more properties:

• **Commutation with conversion**

If $(T,X) \in \mathbf{T}(X)$, we let $v(T,X)$ be its converse, defined by $v(T,X) = \{(a,b) \in X \times X : (b,a) \in (T,X)\}$.

A ranking rule f commutes with conversion if
for all tournaments $(T,X) \in \mathbf{T}(X)$: $[f(v(T,X)) = vf(T,X)]$.

If f satisfies this condition, application of f to the reversed tournament, reverses the original ranking. For example, f_{trans} and f_{out} commute with conversion. As may be verified, using the results from table 3.1, f_{cop} does not satisfy this condition.

It is easy to verify that f_{out} commutes with permutations, conversion and concatenation. These three conditions together with an independence condition which we will introduce hereafter, are necessary and sufficient in order that a map f, which assigns a weak ordering on X to a tournament on X, satisfies $f = f_{out}$.

• **Partial independence**

Suppose we take two elements a and c, and change relations between alternatives, not involving a or c. Only change, for example, relations inside a set B, to which a and c do not belong. Then f is said to be partial independent, if the relative ranking of a and c is not disturbed by of changes in B.

It may be easily verified that the ranking rule f_{out} is partial independent. Indeed, the number of comparisons in which alternative a prevailed, is the same as before. This also is true for alternative c. Remarkably enough, partial independence, together with the three elementary commuting properties, fully characterize f_{out}, see theorem 3.5. In order to present this theorem, we more precisely introduce the partial independence condition, as was done in Delver, Monsuur and Storcken (1991):

A ranking rule h on $\mathbf{T}$ *is partial independent, if and only if for all* $X \subseteq U$ *and all tournaments* (T,X) *and* (T',X) *in* $\mathbf{T}(X)$ *and all* $B \subseteq X$:
if $T \cap ((X \times X)-(B \times B)) = T' \cap ((X \times X)-(B \times B))$, *then*
$h(T,X)|(X-B) = h(T',X)|(X-B)$.

To explain this condition, let (T,X), (T',X) $\in$ $\mathbb{T}$(X) only differ on B $\subseteq$ X. So T $\cap$ ((X×X)-(B×B)) = T' $\cap$ ((X×X)-(B×B)). Then partial independence means, that the order of the elements in X-B by h(T,X) and h(T',X) are the same. So in that case the order of the elements in X-B only depends on the order of the elements in T $\cap$ ((X×X)-(B×B)).

Note that f_{out} is partial independent, since if T $\cap$ ((X×X)-(B×B)) = T' $\cap$ ((X×X)-(B×B)) then s_y(T,X) = s_y(T',X) for all y $\in$ X-B. So partial independence as well as commutation with permutations, conversion and concatenation are necessary conditions if $h = f_{out}$.

THEOREM 3.5 (Characterization of f_{out}) *Suppose h is a ranking rule on* $\mathbb{T}$.
Then $h = f_{out}$,
if and only if h is partial independent and
h commutes with permutations, conversion and concatenation.

Proof First of all, we will prove that if h satisfies these three conditions, then for a special type of tournament (T,X), we have that h(T,X) = f_{out}(T,X).

LEMMA *Let h satisfy the conditions of the theorem.*
Let $X = \{a_1, a_2, \ldots, a_n, x\}$, $s \in \{1,\ldots,n-1\}$ *and* $(T,X) \in \mathbb{T}(X)$ *be such that*

$a_i T a_j$ *iff* $i < j$,
$a_i T x$ *iff* $i > s$, *and*
$x T a_i$ *iff* $i \leq s$.

Then $h(T,X) = f_{out}(T,X)$.

Proof Let h and (T,X) be as above. By induction on n $\geq$ 2 we prove that h(T,X) = f_{out}(T,X).
Basis. If n = 2, then (T,X) is a simple 3-circuit, hence h(T,X) = X. Of course f_{out}(T,X) = X, which establishes the equality.
Induction step. Suppose n > 2. Let B = $\{x, a_1\}$.
Note that (T,X) $\cap$ ((X×X)-(B×B)) = ($\{a_1\}$ » (T,X-$\{a_1\}$)) $\cap$ ((X×X)-(B×B)).
Hence by the partial independency of h, we have:

h(T,X)|X-B = ($\{a_1\}$ » h(T,X-$\{a_1\}$))|(X-B)

$$= (\{a_1\} \gg f_{out}(T,X-\{a_1\}))|(X-B)$$
$$= \{a_2\} \gg \ldots \gg \{a_{s-1}\} \gg \{a_s,a_{s+1}\} \gg \{a_{s+2}\} \gg \ldots \gg \{a_n\}. \quad (1)$$

Similarly it follows that

$$h(T,X)|X-\{x,a_n\} = \{a_1\} \gg \ldots \gg \{a_s,a_{s+1}\} \gg \ldots \gg \{a_{n-1}\}. \quad (2)$$

Since $h(T,X) \in \mathbb{W}(X)$ and $n > 2$, it follows from (1) and (2) that

$$\begin{aligned} h(T,X)|X-\{x\} &= \{a_1\} \gg \{a_2\} \gg \ldots \gg \{a_s,a_{s+1}\} \gg \ldots \gg \{a_n\} \\ &= f_{out}(T,X)|(X-\{x\}). \end{aligned}$$

Take $G = \{a_s,a_{s+1},\ldots,a_n\}$ and $D = \{a_1,\ldots,a_{s+1}\}$.
$(T,X) \cap (X-G)^2 = ((T,X-\{a_s\}) \gg \{a_s\}) \cap (X-G)^2$. Hence by the partial independency of h it follows that

$$\begin{aligned} h(T,X)|(X-G) &= h((T,X-\{a_s\}) \gg \{a_s\})|(X-G) \\ &= (h(T,X-\{a_s\}) \gg \{a_s\})|(X-G) && \text{(concatenation)} \\ &= h(T,X-\{a_s\})|(X-G) \\ &= f_{out}(T,X-\{a_s\})|(X-G) && \text{(induction)} \\ &= f_{out}(T,X)|(X-G) && \text{(concatenation and partial independence)} \quad (3) \end{aligned}$$

Similarly it follows that $h(T,X)|(X-D) = f_{out}(T,X)|(X-D)$. (4)
So it is sufficient to prove that:

$(x,a_s) \in h(T,X)$ iff $(x,a_s) \in f_{out}(T,X)$ and (5)

$(x,a_{s+1}) \in h(R_X)$ iff $(x,a_{s+1}) \in f_{out}(T,X)$. (6)

Note that $h(T,X) \in \mathbb{W}(X)$. Hence this holds if there is an $a_i \in \{a_1,\ldots,a_{s-1}\}$ such that $(x,a_i) \in h(T,X)$ or there is an $a_j \in \{a_{s+2},\ldots,a_n\}$ such that $(a_j,x) \in h(T,X)$.
Suppose $(a_i,x) \in ah(T,X)$ for all $a_i \in \{a_1,\ldots,a_{s-1}\}$ and $(x,a_j) \in ah(T,X)$ for all $a_j \in \{a_{s+2},\ldots,a_n\}$. It is sufficient to prove (5) and (6) for this case. Now by (3) and (4) it follows that for the Copeland scores we have $s_x(T) = s_{a_s}(T) = s_{a_{s+1}}(T) = s$ and $2s = n$. Take a permutation σ, such that $\sigma(a_i) = a_{n-i+1}$ for all $i \in \{1,\ldots,n\}$. Note that $\sigma(T,X) = v(T,X)$.

Hence $(x,a_s) \in h(T,X)$ iff $(a_s,x) \in h(v(T,X))$
iff $(a_s,x) \in h(\sigma(T,X))$
iff $(a_s,x) \in \sigma h(T,X)$
iff $(a_{s+1},x) \in h(T,X)$.

Hence $\{x,a_s,a_{s+1}\}^2 \subseteq h(T,X)$ and h satisfies (5) and (6). ■

By virtue of this lemma we can prove the theorem:

The "only if" part is obvious. We prove the "if" part. Suppose h satisfies those three conditions. Let $(T,X) \in \mathbb{T}(X)$. Take $(x,y) \in X{\times}X$. It is sufficient to prove that $h(T,X)|\{x,y\} = f_{out}(T,X)|\{x,y\}$.

Let better(a,T) := $\{b \in X : (b,a) \in (T,X)\}$,
worse(a,T) := $\{b \in X : (a,b) \in (T,X)\}$, for $a \in X$.

Take X_1 = better(x,T) $\cap$ better(y,T),
X_2 = better(x,T) $\cap$ worse(y,T),
X_3 = worse(x,T) $\cap$ better(y,T) and
X_4 = worse(x,T) $\cap$ worse(y,T). See figure 3.1.

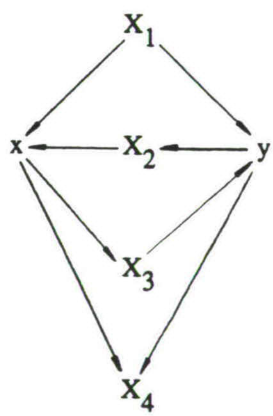


Fig. 3.1

Take $(T_2,X_2) \in \mathbb{L}(X_2)$ and $(T_3,X_3) \in \mathbb{L}(X_3)$ arbitrarily.

Let $Y = X_2 \cup X_3 \cup \{x,y\}$.

Define $(T',Y) =$

$$\{(a,b) \in Y \times Y : (a,b) \in (T_2,X_2) \gg (T_3,X_3), \\ (a,b) \in X_2 \times \{x\} \cup \{y\} \times X_2 \text{ or} \\ (a,b) \in X_3 \times \{y\} \cup \{x\} \times X_3\}.$$

By the lemma we have $f_{out}(T',Y) = h(T',Y)$. Note that $(T,X) \cap (X-(X-\{x,y\}))^2 = ((T,X_1) \gg (T',Y) \gg (T,X_4)) \cap (X-(X-\{x,y\}))^2$.

So by the partial independency we have:

$$h(T,X)|\{x,y\} = h(T,X_1) \gg (T',Y) \gg (T,X_4))|\{x,y\}.$$

Using the commutation with concatenation, we have:

$h(T,X)|\{x,y\} = (h(T,X_1) \gg h(T',Y) \gg h(T,X_4))|\{x,y\}$. Hence
$h(T,X)|\{x,y\} = h(T',Y)|\{x,y\} = f_{out}(T',Y)|\{x,y\}$. ■

Note that f_{cop} does not verify the partial independence condition: take a cyclic tournament on four alternatives and reverse the preference between the two alternatives having the highest score.

3.5.3 Δ-Independence of irrelevant alternatives (Δ-IIA)

One may wonder why we do not require the commutation with restriction:

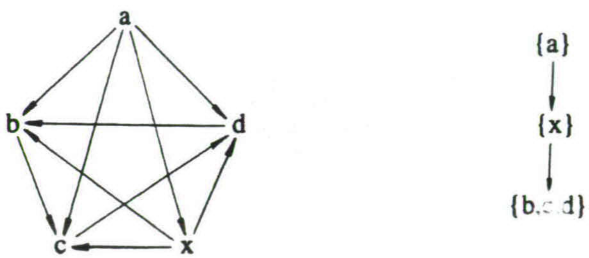
$$f(T,Y) = f(T,X)|Y, \text{ for all } Y \subseteq X.$$

This condition means that the image of (T,X) in Y, only depends on (T,Y). So it guarantees a kind of Independence of Irrelevant Alternatives. See Arrow (1978) and (1948). That this is a strong condition, is illustrated by the fact that when in addition f commutes with concatenation, then f would map every strict preference onto the same strict preference: $f(T,X)|\{a,b\} = f(T,\{a,b\}) = \{a\} \gg \{b\}$ if $aTb : (T,X)$, hence $f(T,X) = (T,X)$. Since a tournament is not in general a weak order, commuting with restriction and concatenation simultaneously, lead to nonexistence of f.

We therefore will consider weaker forms of commutation with restriction, so-called independence conditions concerning variable domains. An example is the right-interval consistency. Another example is provided by the following condition.

• **Δ-IIA**

In Delver, Monsuur and Storcken (1991) a weaker form of commutation with restriction condition was introduced: the Δ-*I*ndependence of *I*rrelevant *A*lternatives condition (Δ-IIA). Before giving its formal definition, let us illustrate it using the tournament in figure 3.2. Let a ranking rule f be Δ-IIA, and assume that with x deleted from the tournament, it would produce the ranking {a} followed by {b,c,d}. In direct comparisons a dominates x, which in turn dominates b,c and d. According to the Δ-IIA condition, f applied to the full tournament would then also place x between a and the cycle {b,c,d}.



tournament ranking

Figure 3.2

In this small tournament f_{trans}, f_{out} and f^{*}_{out} all satisfy the Δ-IIA condition. The f_{trans} algorithm does so in general, but this is not the case for f_{out} and all its refinements.

We now formally introduce the Δ-IIA condition. Let a(R,X) be the asymmetric part of a relation (R,X). We define upp(x,T) = {y ∈ X: yTx} and low(x,T) = {y ∈ X: xTy}. Finally, $X_x = X\backslash\{x\}$.

A ranking rule f on $\mathbf{T}$ *is called Δ-IIA iff for all* $(T,X) \in \mathbf{T}$ *and* $x \in X$*:*

$$\forall u \in upp(x,T)\ \forall w \in low(x,T)[(u,w) \in af(T,X_x)] \Rightarrow$$
$$f(T,X) = f(T,X_x)|upp(x,T) \text{ » } \{x\} \text{ » } f(T,X_x)|low(x,T).$$

Unfortunately, f_{out} and f^*_{out} both violate the Δ-IIA condition. Consider the following example.

T_3	a	x	b	c	d	e	f	g
a	-	1	1	1	1	1	0	0
x	0	1	1	1	1	1	1	1
b	0	0	-	1	1	0	1	0
c	0	0	0	-	0	1	1	1
d	0	0	0	1	-	0	0	1
e	0	0	1	0	1	-	0	1
f	1	0	0	0	1	1	-	0
g	1	0	1	0	0	0	1	-

Then $f_{out}(T_3,X) = \{x\} \gg \{a\} \gg \{b,c,e,f,g\} \gg \{d\}$. If we consider the tournament (T_3,X_x), we obtain $f_{out}(T_3,X_x) = \{a\} \gg \{b,c,e,f,g\} \gg \{d\}$. Now, $upp(x,T_3) = \{a\}$, $low(x,T_3) = \{b,c,d,e,f,g\}$, and

$\forall u \in upp(x,T_3)\ \forall w \in low(x,T_3)\ [(u,w) \in af_{out}(T_3,X_x)]$.

So, in (T_3,X_x), we inserted x between the alternative a and the rest, resulting in the tournament (T_3,X). According to the Δ-IIA condition, we have to obtain the ranking

$\{a\} \gg \{x\} \gg \{b,c,e,f,g\} \gg \{d\}$

which is not the case.

Note that this example also shows that every refinement of f_{out}, including f^*_{out}, violates the Δ-IIA condition.

Compare this condition with the corresponding Δ-IIA condition for choice functions, introduced in chapter 2, section 2.7.1. If S is Δ-IIA, then f_S is Δ-IIA. It is not clear yet, whether or not the converse assertion is true.

The following two theorems may be found in Delver et al. (1991).

THEOREM 3.6 *The ranking rule f_{trans} is Δ-IIA.*

Proof Let $T \in \mathbb{T}(X)$. Suppose that $u \gg w : f_{trans}(T,X_x)$. This is equivalent to $u(a\tau(T,X_x))w$. Inserting x between the indifference classes of u and w in the ranking $f_{trans}(T,X_x)$, we obtain $u(a\tau T)w$. ■

As one can see in the example above, f_{out} satisfies a weaker Δ-IIA condition.

A ranking rule f on $\mathbf{T}$ *is called weakly-Δ-IIA iff for all* $(T,X) \in \mathbf{T}$ *and* $x \in X$: $\forall u \in upp(x,T)\ \forall w \in low(x,T)[(u,w) \in af(T,X_x)] \Rightarrow$

$$f(T,X)\,|\,X_x = f(T,X_x).$$

It means that 'properly' inserting an alternative x does not change the relative ranking of the original alternatives.

THEOREM 3.7 *The ranking rules* f^*_{out} *and* f_{out} *are weakly-Δ-IIA.*

Proof Since $s_u(T,X_x) + 1 = s_u(T,X)$ for all $u \in upp(x,T)$ and $s_w(T,X_x) = s_w(T)$ for all $w \in low(x,T)$, the proof follows easily. ■

In chapter 2, section 2.7.1, we showed that the uncovered set, given by the choice function *uc*, is not Δ-IIA. The rule f_{uc} is weakly-Δ-IIA:

PROPOSITION 3.8 *The ranking rule* f_{uc}, *induced by the choice function uc, is weakly-Δ-IIA.*

Proof We show that the following is true.
For all $T \in \mathbf{T}(X)$, for all $x \in X$:
if for all $a \in uc(T,X_x)$: aTx, then $uc(T)$ is equal to either $uc(T,X_x)$ or $uc(T,X_x) \cup \{x\}$.
For this, it is sufficient to prove that for all $b \in X_x$ the following holds: b is covered in (T,X_x) iff b is covered in T.
(if) Evident. (only if) Since the cover relation is transitive, the alternative b is covered by an alternative $a \in uc(T,X_x)$. We prove that b is covered by a in T. Suppose a does not cover b in T, hence b can reach a in two steps. Then we must have: bTx while xTa. But this is impossible, because $a \in uc(T,X_x)$. ■

3.5.4 The score vector as a sufficient statistic

A ranking may be seen as a synthesis or balancing of the binary decisions contained in cycles. From this point of view, an interesting condition might be:

- **Independence of 3-cycle orientation**

  A ranking rule f [choice function S] *on* $\mathbb{T}(X)$ *is said to be independent of 3-cycle orientation if for all pairs of tournaments (T,X), (T',X)* $\in \mathbb{T}(X)$,

  if (T',X) can be obtained from (T,X) by reversing the orientation of a 3-cycle, then f(T,X) = f(T',X) [S(T,X) = S(T',X)].

In balancing information contained in cycles, a ranking rule f satisfying this independence condition, does not take into account the direction of a particular 3-cycle.

One may have some difficulty with this independence condition. For example, an alternative beating a strong alternative cannot be rewarded. Because if we reverse a cycle containing the comparison between these two alternatives, according to the independence condition, nothing will change. This point will be taken up again in section 3.8.

Taken together with just one commutation property, it severely restricts the possible choice of a ranking rule, see theorem 3.9 and corollary 3.10.

In theorem 3.9, we use the term sufficient statistic. There, it means that the ranking only depends upon the scorevector.
Henriet (1985) proved the following two results in the case of ranking complete binary relations. We give alternative proofs for the case of tournaments:

THEOREM 3.9 *Let* $f : \mathbb{T}(X) \to \mathbb{W}(X)$ *be a ranking rule. Then f is independent of 3-cycle orientation, iff*
the score vector is a sufficient statistic in the determination of the ranking produced by f.

Proof (only if) Suppose that (T,X) and (T',X) do have the same score vector. As will be derived in the proof of theorem 5.11 of chapter 5, (T',X) can be obtained from (T,X) by a sequence of 3-cycle reversals. Hence, f(T,X) = f(T',X). (if) Evident. ∎

COROLLARY 3.10 *Suppose that the ranking rule f commutes with permutation and is independent of 3-cycle orientation. Let (T,X) be a tournament and assume that for two alternatives x and y, we have $s_x = s_y$. Then $x \approx y : f(T,X)$.*

Proof Let $X = \{a_1,\dots,a_n\}$. Take two permutations σ_1 and σ_2, such that

$$s_{a_i}(\sigma_k T) \geq s_{a_j}(\sigma_k T) \text{ if } i \leq j,\ k = 1,2.$$

Since $s_x = s_y$, we may take $\sigma_1 x = a_p$, $\sigma_1 y = a_{p+1}$ and $\sigma_2 x = a_{p+1}$, $\sigma_2 y = a_p$, for some $p \in \{1,\dots,n\}$.
Because f is independent of 3-cycle orientation, $f(\sigma_1 T) = f(\sigma_2 T) := W$.
Now, assume that $x \gg y : f(T)$. Then $\sigma_1 x \gg \sigma_1 y : \sigma_1 f(T) = f(\sigma_1 T) = W$, because f commutes with permutation. Further, $\sigma_2 x \gg \sigma_2 y : \sigma_2 f(T) = f(\sigma_2 T) = W$. So, on the one hand, we have $a_p \gg a_{p+1} : W$, on the other hand we have $a_{p+1} \gg a_p : W$. This gives a contradiction. ■

As may easily be verified, the rules f_{out} and f_{trans} do satisfy the conditions of the theorem above. This means that there does exist a function $g : \mathbb{R}^n \to \mathbb{W}(X)$, $n = |X|$, such that $f_{trans}(T) = g(s_1,\dots,s_n)$. This is a well-known result, see proposition 1.3. The rules f_{cop} and f^*_{out} do not satisfy this condition. This may be seen by taking T_4 and using corollary 3.10:

$s_c = s_d$, but $c \gg d : f_{cop}(T_4)$ and $c \gg d : f^*_{out}(T_4)$.

From theorem 3.9 we know that if a ranking rule f is independent of 3-cycle orientation, then the score vector is a sufficient statistic in the determination of the ranking. If we add the condition of weakly Δ-IIA and the commutation with permutation, we can prove more, see theorem 3.11.

In the proof of theorem 3.11, we need to add alternatives. So, instead of ranking rules on $\mathbb{T}$, we have to consider ranking rules on $\mathbb{T}^*$.

The set $\mathbb{T}^*$

The set $\mathbb{T}^*$ is the collection of $\mathbb{T}(X)$, for arbitrary non-empty finite $X = \{x_1,\dots,x_n\}$.

THEOREM 3.11 *Let f be a ranking rule on* $\mathbb{T}^*$. *Suppose that f commutes with permutation, is independent of 3-cycle orientation and satisfies the weak-Δ-IIA condition. Then for each couple (x,y) of alternatives from* $X \subseteq U$ *and all tournaments* $T \in \mathbb{T}(X)$,
if $s_x(T,X) \geq s_y(T,X)$ *then* $x \geq y : f(T,X)$.

Proof From corollary 3.10, we know that the first two condition do imply that $a \approx b : f(T)$ whenever $s_a = s_b$. Now, suppose that $s_a > s_b$, but $b \gg a : f(T)$. Thus we may partition X into two subsets A and B, where $a \in A$ and $b \in B$, such that for all $u \in B$ and all $v \in A$, $u \gg v : f(T)$. Next, introduce new alternatives $x_1, ..., x_p$, where $p = s_a - s_b$. Insert them between B and A: $uT'x_i$, $x_iT'v$ for all $u \in B$, $v \in A$, $i \in \{1,...,p\}$. Then, using the weak-Δ-IIA condition p times, we obtain $b \gg a : f(T')$. But, since $s_a(T') = s_b(T')$, $b \gg a : f(T')$ is impossible. ∎

Using this theorem, we can characterize the Copeland choice function, which was introduced in chapter 2. In the formulation of the theorem, we assume that $S(T,X) = \text{Best}(f(T,X))$, for a ranking rule f. This is no restriction, since $S(T,X) = \text{Best}(f_S(T,X))$, where f_S is the S-induced ranking rule.

THEOREM 3.12 (Characterization of the Copeland choice function) *The choice function cop on* $\mathbb{T}^*$ *equals Best(f) for a ranking rule f satisfying the following conditions. The rule f commutes with permutation, is independent of 3-cycle orientation and is weakly-Δ-IIA. Moreover, if a choice function S = Best(f) on* $\mathbb{T}^*$, *where f satisfies these conditions, then for all tournaments* $(T,X) \in \mathbb{T}^*$,
we have that $cop(T,X) \subseteq S(T,X)$.

Proof The verification that *cop* satisfies the conditions mentioned is straightforward, if we note that $cop = \text{Best}(f_{out})$. We only prove the 'moreover' part. Let $(T,X) \in \mathbb{T}^*$. Suppose that we have two alternatives a and b such that $s_a > s_b$, and $b \in S(T,X) = \text{Best}(f)$. Using theorem 3.11, we obtain $a \in S(T,X)$. So, $cop(T,X) \subseteq S(T,X)$. ∎

Logical independence of the conditions of theorem 3.12.
(1) $cop^* = \text{Best}(f^*_{out})$, where f^*_{out} is weakly-Δ-IIA. But it is not

independent of 3-cycle orientation.

(2) $vcop$ = Best(vf_{out}), where vf_{out}, the converse of f_{out}, is independent of 3-cycle orientation. But, as showed in theorem 3.11, it is not weakly-Δ-IIA.

Note that in the proof above, we need to express S in terms of Best(f). Indeed, a choice function S being weakly-Δ-IIA would mean something like:

$$\text{if } aTx \text{ for all } a \in S(T,X) \text{ then } S(T,X\cup\{x\}) = \begin{cases} S(T,X) & \text{or} \\ S(T,X)\cup\{x\} & \text{or} \\ \{x\} \end{cases}.$$

This last possibility, $S(T,X\cup\{x\}) = \{x\}$ gives problems in the proof of theorem 3.12.

See Henriet (1985) for characterizations of cop on arbitrary relations, that use a monotonicity argument instead of the weak-Δ-IIA condition.

We consider what will happen, if we replace the weak-Δ-IIA condition by the Δ-IIA condition. For that purpose, we introduce the choice function cop^{δ}.

The choice function $\mathbf{cop}^{\delta}$

Suppose $(T,X) \in \mathbf{T}^*$. Let $f_{out}(T,X) = X_1 \gg \ldots \gg X_n$. Then $cop^{\delta}(T,X) := \bigcup_{i=1}^{j} X_i$, where $j \geq 1$ is the smallest number such that

for all $y \in \bigcup_{i=1}^{j} X_i$, we have $s_y \geq |X \setminus \bigcup_{i=1}^{j} X_i|$.

The reason for taking these union of indifference classes of f_{out}, is that this choice function cop^{δ} is Δ-IIA (introduced in chapter 2). The verification of this is straightforward.

We have the following characterization.

THEOREM 3.13 (Characterization of cop^{δ}). *The choice function cop^{δ} on $\mathbf{T}^*$ satisfies neutrality, independence of 3-cycle orientation and Δ-IIA. Moreover, if a choice function S on $\mathbf{T}^*$ satisfies these conditions, then for all tournaments $(T,X) \in \mathbf{T}^*$, we have that $cop^{\delta}(T,X) \subseteq S(T,X)$.*

Proof We only give a proof of the 'moreover' part. Let $(T,X) \in \mathbf{T}^*$, and

$cop^{\delta}(T,X) = \bigcup_{i=1}^{j} X_i$, where $X_1 \gg \ldots \gg X_n = f_{out}(T,X)$. Analogous to the proof of corollary 3.10, we may deduce that if $a \in S(T,X)$ and $s_a = s_b$, then $b \in S(T,X)$. Next, suppose that $s_a > s_b$, $b \in S(T,X)$, while $a \notin S(T,X)$. We derive a contradiction. To this end, introduce p new alternatives $y_1,\ldots,y_p$, $p = s_a - s_b$. Let $(T,X\cup\{y_1,\ldots,y_p\})$ be such that for all $i \in \{1,\ldots,p\}$, for all $u \in S(T,X)$ and all $v \in X\backslash S(T,X)$, we have: uTy_i and y_iTv. Using the Δ-IIA condition p times, we obtain that $b \in S(T,X\cup\{y_1,\ldots,y_p\})$. But now, a and b do have the same score, so $a \in S(T,X\cup\{y_1,\ldots,y_p\})$, which is impossible, because S is Δ-IIA.

This means that $S(T,X) = \bigcup_{i=1}^{k} X_i$, for some number k.

Next, suppose that k is smaller that the number j. Then there is an alternative $z \in S(T,X)$, such that $s_z < |X\backslash S(T,X)|$. Consider $(T,X\cup\{\alpha\})$, where for all $u \in S(T,X)$ and all $v \in X\backslash S(T,X)$, $uT\alpha$ and αTv. Since, because S is Δ-IIA, $S(T,X) = S(T,X\cup\{\alpha\})$, we obtain $z \in S(T,X\cup\{\alpha\})$. But, in the tournament $(T,X\cup\{\alpha\})$, the score of α is at least as large as the score of z. Thus $\alpha \in S(T,X\cup\{\alpha\})$, which contradicts the assumption of S being Δ-IIA.■

Logical independence of the conditions in theorem 3.13.

(1) The choice function *strat* is Δ-IIA, but it is not independent of 3-cycle orientation.

(2) The choice function *cop* is independent of 3-cycle orientation, but it violates the Δ-IIA condition.

3.5.5 A survey of characterizations in the literature

Another characterization of f_{out} may be found in Rubinstein (1980). We present his theorem.

- **Positive responsiveness**

Let (T,X) be a tournament on X and $x,y \in X$, $x \neq y$. Then yx(T,X) is a tournament that is defined by

$$yx(T,X) = \{(y,x)\} \cup (T,X)\backslash\{(x,y)\}.$$

Let f be a ranking rule. Suppose (T,X) is a tournament on X and x,y are two distinct alternatives. Assume $x \geq y : f(T,X)$. Let z be a third alternative such that zTx. Then f is said to satisfy positive responsiveness if $x \gg y : f(xzT,X)$.

THEOREM 3.14 *(Rubinstein, 1980) The Copeland ranking rule f_{out} is the only ranking rule that satisfies commutation with permutation, positive responsiveness and partial independence.* ∎

Henriet (1985) gives a generalization of Copeland scores in a complete binary relation R. To each couple (x,y) of alternatives an integer value R(x,y) is associated, defined by

$$R(x,y) = 2 \text{ if } xRy \text{ and } \neg yRx$$
$$R(x,y) = 1 \text{ if } xRy \text{ and } yRx$$
$$R(x,y) = 0 \text{ if } \neg xRy \text{ and } yRx.$$

Then the Copeland score of some alternative a is equal to

$$C(a) = \sum_{b \in X,\ b \neq a} R(a,b).$$

The corresponding rule is defined by

$$a \geq b : f_C(R,X) \text{ iff } C(a) \geq C(b).$$

In Bouyssou (1992), the relation R is replaced by a valued binary relation R. To each couple (a,b), $a \neq b$, a value $R(a,b) \in [0,1]$ is assigned. Then the Net Flow Method is introduced, which is based upon the score

$$S_{NF}(a) = \sum_{b \in X \setminus \{a\}} (R(a,b) - R(b,a)).$$

Henriet and Bouyssou give characterizations for their rules. Compared with the characterization of f_{out}, given by Rubinstein, the partial independence is replaced by a notion of independence of cycles. For tournaments, the condition of Henriet is the same as our independence of 3-cycle orientation.

Using theorem 3.9, it is easy to prove that the conditions given by Henriet also characterize f_{out}:

THEOREM 3.15 *Let f be a ranking rule on $\mathcal{T}$. Then $f = f_{out}$ iff f commutes with permutation, satisfies positive responsiveness and is independent of 3-cycle orientation.* ■

3.6 RANKING RULES INDUCED BY NEAREST LINEAR ORDERS

In this section, we introduce three ranking rules based on nearest adjoining (linear) orders.

We refer to chapter 2, section 2.5.4 for the definition of nearest adjoining orders. We shall define three ranking rules, that are based upon these orders.

f_{slat1}

Suppose $T \in \mathcal{T}(X)$. Let $sl(T)$ be the set of elements that are best in a nearest adjoining order (*sl* stands for Slater, 1961).

We define the rule f_{slat1} on $\mathcal{T}$ as follows. Let $(T,X) \in \mathcal{T}(X)$.

if $X \setminus sl(T,X) = \emptyset$ then $f_{slat1}(T,X) = sl(T,X)$, else

$f_{slat1}(T,X) = sl(T,X) \gg f_{slat1}(T,X \setminus sl(T,X))$.

It is the *sl*-induced ranking rule.

f_{slat2}

The ranking rule f_{slat2} orders the alternatives to their highest ranking in any nearest adjoining order. For example, in table 3.2, for each of the alternatives d, e and f, there is a nearest adjoining order in which they are ranked fourth, see section 2.5.4, chapter 2.

f_{slat3}

Let $T \in \mathcal{T}(X)$. Define the relation $S \subseteq X \times X$ as follows:

$$S = \bigcup_{L \in N(T)} L.$$

Now we may define the following ranking rule.

$f_{slat3}(T) = \tau S$, τ is the transitive closure.

This means that $x \geq y : f_{slat3}(T)$ iff there exists a sequence of alternatives $x = a_0,\dots,a_m = y$, $m \geq 1$, and a sequence of nearest adjoining orders $L_0,\dots,L_{m-1}$, such that for all $j = 0,\dots,m-1$, we have: $a_j L_j a_{j+1}$.

T_2	a	b	c	d	e	f	g	h
a	-	1	1	1	1	1	0	0
b	0	-	1	1	1	1	0	1
c	0	0	-	1	1	1	1	1
d	0	0	0	-	1	0	1	1
e	0	0	0	0	-	1	1	1
f	0	0	0	1	0	-	1	0
g	1	1	0	0	0	0	-	1
h	1	0	0	0	0	1	0	-

f_{slat1}	f_{slat2}	f_{slat3}
{a}	{a}	{a}
{b}	{b}	{b}
{c}	{c}	{c}
{d,e,f}	{d,e,f}	{d,e,f,g,h}
{g}	{g,h}	
{h}		

Table 3.2 Rankings of T_2.

As an illustration, we take the tournament from table 3.2. For example, although in all nearest adjoining orders L, eLh, we have that $e \approx h : f_{slat3}(T)$. This is true because we have the sequence h, f, e and two nearest adjoining orders:
$a \gg b \gg c \gg d \gg e \gg h \gg f \gg g$ and $a \gg b \gg c \gg f \gg d \gg e \gg g \gg h$. In the first one $h \gg f$, in the second one $f \gg e$. Hence $h \geq e : f_{slat3}(T)$. On the other hand, because $e \gg h$ in all nearest adjoining orders, we also have $e \geq h : f_{slat3}(T)$.

The rules f_{slat1} and f_{slat2} are not Δ-IIA. This may be seen by using T_7, given in the following section. $\mathrm{Best}(f_{slat1}(T_7)) = \mathrm{Best}(f_{slat2}(T_7)) = \{a,b,c\}$. If we insert X between {a,b,c} and {d,e,f,g}, we obtain T_7'. Now, $\mathrm{Best}(f_{slat1}(T_7')) = \mathrm{Best}(f_{slat2}(T_7')) = \{a\}$, because there is just one nearest adjoining order: $a \gg b \gg c \gg d \gg e \gg f \gg g$.

THEOREM 3.16 *The ranking rule f_{slat3} is Δ-IIA.*

Proof Suppose that we need m reversals to convert (T,X_x) into a linear order. Let $f_{slat3}(T,X_x) = X_1 \gg \dots \gg X_n$. Now we insert x between X_i and X_{i+1}, as described in the Δ-IIA condition, giving the tournament (T,X). We

have to prove that $f_{slat3}(T,X) = X_1 \gg \ldots \gg X_i \gg \{x\} \gg X_{i+1} \gg \ldots \gg X_n$.
By construction of the rule f_{slat3}, we know that both $upp(x,T) = X_1 \cup \ldots \cup X_i$ and $low(x,T) = X_{i+1} \cup \ldots \cup X_n$ are intervals in all nearest adjoining orders of (T,X_x).
It is sufficient to prove that all nearest adjoining orders of (T,X) can be obtained from the nearest adjoining orders of (T,X_x), by inserting x at the appropriate position: between upp(x,T) and low(x,T).
Take a linear order L on X. If upp(x,T) or low(x,T) are no intervals in $L|X_x$, it is clear that we need more than m reversals to convert (T,X) into L. Now suppose that a linear order is obtained by inserting x in a nearest adjoining order of (T,X_x), but not at the right position. Then again, we need more than m reversals. ■

3.7 LOCAL AND GLOBAL DIFFERENCES

In this section, we discuss two independence conditions. They differ in just one aspect. The first condition is focused on the binary comparisons at the local level of analysis, the second one on the global level. If, in a binary relation, the alternatives from a set B behave like one alternative with respect to alternatives from X\B, hence are interchangeable, these conditions state that we may reduce the set of available alternatives to B, without the need to change the relative ranking of these alternatives.

3.7.1 Introduction of the two conditions

If a tournament (T,B) is separable from (T,X) (see section 2.7.1), the alternatives of B behave like a single alternative. Each alternative x from X\B delivers the same local information to all b ∈ B. So, the local differences between b_1 and $b_2 \in B$ are reduced to their local differences inside B. If we concentrate on local differences, the following condition may be significant.

• **Separability**

We say that a ranking rule f on $\mathbf{T}$ respects separable tournaments, if for all nonempty subsets $X \subseteq U$, for all $(T,X) \in \mathbf{T}(X)$ and for all tournament $(T,B) \in \mathbf{T}(B)$ that are separable from (T,X):

$$f(T,X)|B = f(T,B).$$

Consider the circular tournament T_1 on 4 alternatives. As may be easily verified, $(T_1,\{b,c\})$ is separable from (T_1,X).
But, $f_{trans}(T_1,\{b,c\}) = b \gg c$ while $b \cong c : f_{trans}(T_1,X)|\{b,c\}$. So, the rule f_{trans} does not satisfy this condition. This is not surprising, if we realize that f_{trans} is based on global considerations, while the condition is concerned with local, binary information, as explained above.

• **Interval consistency**

Let $W \in \mathbf{W}(X)$. $I \subseteq X$ is an *interval* of W if for all $x,y \in I$ and $z \in X$ if (x,z), $(z,y) \in W$, then $z \in I$. Hence if $X_1 \gg X_2 \gg .. \gg X_n = W$ then every interval I is a union of successive indifference classes of W.

Consider the following tournament T_5 on $X = \{a,b,c,d,e,f\}$.

T_5	a	b	c	d	e	f
a	-	0	1	1	1	1
b	1	-	0	1	1	1
c	0	1	-	0	1	1
d	0	0	1	-	0	1
e	0	0	0	1	-	0
f	0	0	0	0	1	-

Then $f_{out}(T_5) = \{a,b\} \gg \{c\} \gg \{d\} \gg \{e,f\}$. Furthermore $I = \{c,d\}$ is an interval of $f_{out}(T_5)$. Now, consider the ranking of the tournament restricted to I: $f_{out}(T_5,\{c,d\}) = \{d\} \gg \{c\}$. We see that the rule f_{out} changes existing intervals. This means that it is profitable for alternative d to adhere to a more local or binary level of analysis. This means that the tension between the local and global level becomes stronger. In this respect, the rule f_{out} is not very stable. We introduce for arbitrary ranking rules the condition of interval consistency:

A ranking rule f on $\mathbf{T}$ is said to be interval consistent if for all $(T,X) \in \mathbf{T}$ and all intervals I of $f(T,X)$:

$$f(T,X)|I \quad = \quad f(T,I).$$

This means that the ranking of alternatives in an interval I only depends upon the tournament restricted to I. So, in that case we have $f \equiv f^*$.
As announced, the condition of interval consistency is a global equivalent of the separability condition. This may be seen by observing that in an interval I, the alternatives behave like one alternative to the alternatives of X\I.

Using $T = T_5$, we may show that f^*_{out} and f_{cop} do not satisfy the condition of interval consistency.
Note that this condition is stronger than the condition of right-interval consistency. So, if a ranking rule f is interval consistent, it is induced by a choice function.

We conclude this subsection with the following proposition:

PROPOSITION 3.17 *If a ranking rule f on $\mathbf{T}$ is interval consistent, commutes with concatenation and is weakly-Δ-IIA, then f is Δ-IIA.*

Proof Suppose $f(T,X_x) = X_1 \gg \dots \gg X_n$. Suppose that in (T,X), the alternative x is inserted between X_i and X_{i+1}. Because f is weakly-Δ-IIA, $f(T,X)|X_x = X_1 \gg \dots \gg X_n$.
Hence the alternatives in $I_1 = X_1 \cup \dots \cup X_i \cup \{x\}$ and/or $I_2 = X_{i+1} \cup \dots \cup X_n \cup \{x\}$ constitute an interval of $f(T,X)$. Suppose I_1 is an interval. Because f is interval consistent, $f(T,X)|I_1 = f(T,I_1)$, which is $X_1 \gg \dots \gg X_i \gg \{x\}$, because f commutes with concatenation. But then I_2 too is an interval, hence $f(T,X)|I_2 = \{x\} \gg X_{i+1} \gg \dots \gg X_n$. ∎

3.7.2 Verifications

In what follows, we verify the two conditions for some ranking rules that were introduced in previous sections.
The following two results are easy to prove.

PROPOSITION 3.18 *The ranking rule f_{trans} is interval consistent.* ∎

PROPOSITION 3.19 *The ranking rules f_{out}, f^*_{out} and f_{cop} respect separable tournaments.* ■

The tournament from table 3.2 shows that f_{slat2} is not interval consistent because of the interval {g,h}. We show that f_{slat1} is not interval consistent either:

T_6	a	b	c	d	e
a	-	1	1	1	0
b	0	-	1	1	1
c	0	0	-	1	0
d	0	0	0	-	1
e	1	0	1	0	-

$f_{slat1}(T_6) = \{a,e\} \gg \{b\} \gg \{c\} \gg \{d\}$.
But $f_{slat1}(T_6,\{a,e\}) = \{e\} \gg \{a\}$.
Note that $f_{slat3}(T_6) = X$.

PROPOSITION 3.20 *The ranking rule f_{slat3} is interval consistent, so it is induced by a choice function.*

Proof

Claim 1. By construction of the ranking rule f_{slat3}, we know the following. If $f_{slat3}(T,X) = Y_1 \gg ... \gg Y_n$, then for all $i < j$, all $a \in Y_i$ and all $b \in Y_j$, we have aLb in any nearest adjoining order. Indeed, suppose bLa, then $b \geq a : f_{slat3}(T,X)$.

Claim 2. Now, suppose that $a_1 \gg ... \gg a_n$ is a nearest adjoining order of (T,X). Then it is easy to verify that:
$b_s \gg ... \gg b_{s+t}$ is a nearest adjoining order of $(T,\{a_s,...,a_{s+t}\})$, where $s \geq 1$ and $s+t \leq n$, iff
$a_1 \gg ... \gg a_{s-1} \gg b_s \gg ... \gg b_{s+t} \gg a_{s+t+1} \gg ... \gg a_n$ is a nearest adjoining order of (T,X).

Now, let $u, v \in I$, an interval of $f_{slat3}(T,X)$, and suppose that we have $u \geq v : f_{slat3}(T,X)$. We have to prove that $u \geq v : f_{slat3}(T,I)$. Because $u \geq v : f_{slat3}(T,X)$, we know that there exists a sequence of alternatives $u = u_0,...,u_m = v$, and a sequence $L_0,...,L_{m-1}$ of nearest adjoining orders of (T,X) such that for all $j = 0,...,m-1$, we have: $u_j L_j u_{j+1}$. Note that all $u_j \in I$. Using claim 1, we know that I is an interval of each L_j where $0 \leq j \leq m-1$. Applying claim 2, where $L_j = a_1 \gg a_2 \gg ... \gg a_n$ and I =

$\{a_s,\ldots,a_{s+t}\}$, we see that we obtain a sequence of nearest adjoining orders $L'_j = L_j|I$ for (T,I), in which $u_j L' u_{j+1}$. Hence $u \geq v : f_{slat3}(T,I)$. Using an analogous line of reasoning, we are able to show: if $u \gg v : f_{slat3}(T,X)$ then $u \gg v : f_{slat3}(T,I)$. ∎

The rule f_{slat3} does not respect separable tournaments: Take the tournament $T = T_7$ below. Then $f_{slat3}(T_7) = \{a,b,c,d,e,f,g\}$, because there are three nearest nearest adjoining orders:
$a \gg b \gg c \gg d \gg e \gg f \gg g$, $b \gg c \gg d \gg e \gg f \gg g \gg a$ and
$c \gg d \gg e \gg f \gg g \gg a \gg b$. But $(T_7,\{e,f\})$ is separable from (T,X).

T_7	a	b	c	d	e	f	g
a	-	1	1	1	0	0	0
b	0	-	1	1	1	0	0
c	0	0	-	1	1	1	1
d	0	0	0	-	1	1	1
e	1	0	0	0	-	1	1
f	1	1	0	0	0	-	1
g	1	1	0	0	0	0	-

PROPOSITION 3.21 *The ranking rule f_{slat2} respects separable tournaments.*

Proof Suppose B is a separable subset of X in $T \in \mathcal{T}(X)$. We first prove that there exists a nearest adjoining order such that B is an interval. Take $s,t \in B$ such that s and t are separated by $Y \subseteq X \backslash B$ in a nearest adjoining order (L,X): $s \gg Y \gg t$ is an interval of (L,X). Because B is separable, we know that for each $y \in Y$, either $y \gg B$ or $B \gg y$. Let m^+ denote the number of elements of Y such that $B \gg y$, let m^- be the number of elements of Y such that $y \gg B$. Because (L,X) is a nearest adjoining order, we have $m^+ = m^-$. This means that we may change (L,X) so that s and t are consecutive, while keeping a nearest adjoining order, by moving t towards s or s towards t. Continue this process for other elements from B.
Now the remaining part of the proof may be accomplished with the help of claim 2 in the proof of the previous proposition, by defining $\{a_s,\ldots,a_{s+t}\} = B$ and for each choice of $b \in B$ letting $a_1 \gg \ldots \gg a_n$ be a nearest adjoining order in which b is highest and in which B is an interval. ∎

There do exist ranking rules that satisfy the two conditions of this section.

PROPOSITION 3.22 *The induced ranking rules f_{mc} and f_{strat} satisfy both the conditions of separability and interval consistency.*

Proof As shown in chapter 2, section 2.7.1, the choice functions *mc* and *strat* respect separable tournaments. So, the induced ranking rules respect separable tournaments. Since the ranking rules are induced by choice functions that in addition satisfy the strong superset property (see chapter 2, theorem 2.4 and proposition 2.12), they are interval consistent. ∎

3.8 RATINGS FROM TOURNAMENTS

In this section, we consider two methods to derive ratings from tournaments. We introduce a maximum likelihood procedure and the eigenvector method. After that we give a short discussion on the use of ratings which are derived from ordinal information contained in a tournament.

A maximum likelihood method: f_{prob}

Suppose a system $P = (p_{ab})$, $a,b \in X$ is given, such that for all $a,b \in X$, p_{ab} is the probability that alternative a beats alternative b. The likelihood of a tournament T on X therefore is $L(T) = \prod_{(a,b) \in T} p_{ab}$, where we assume that all choices are made independently of one another.

A pair comparison system $P = (p_{ab})$, $a,b \in X$, satisfies the so-called *strict utility model* if for all x there exist weights $w_x \neq 0$, such that for all a and b, the probability that a beats b is equal to

$$p_{ab} = \frac{w_a}{w_a + w_b}.$$

In that case, we may compute the likelihood of a tournament T given this system P, as follows:

$$L(T) = \prod_{(a,b)\in T} p_{ab} = \frac{\prod w_a^{s_a}}{\prod_{(a,b)\in T}(w_a + w_b)}, \text{ where } (s_a)_{a\in X} \text{ is the score vector.}$$

Now, suppose that we have an irreducible tournament T. We could have generated this tournament using a system P of probabilities p_{ab}, satisfying the strict utility model. We may wonder which values of p_{ab} are the most natural. The strategy we follow is: choose those p_{ab} satisfying the strict utility model, that maximize the function L, giving rise to the *maximum* likelihood estimates p_{ab}.

If the tournament is reducible, it is easy to verify that the maximum value cannot be attained by systems satisfying the strict utility models. The strict utility models do not constitute a compact subset of all systems P. For, suppose that (w_a), $a \in X$ is the point where the maximum value is obtained. Then, if $(T,X) = (T,Y) \gg (T,Z)$, we replace w_z by kw_z, $z \in Z$ and $0 < k < 1$. But now the value of L(T,X) grows, a contradiction.

In case T is irreducible, there exists a unique point where the maximum value is obtained, see David (1988, page 62) or Moon (1968, page 43). As pointed out by Zermelo and Ford, these strengths w give exactly the same ordinal ranking as one would obtain using the scores of the alternatives. See the following proposition, which was presented as an exercise in Moon (1968).

PROPOSITION 3.23 (Zermelo (1929) and Ford (1957)) *Let s_i be the score and let w_i be the maximum likelihood strength of alternative x_i, where $i \in \{1,...,n\}$. If $s_i > (=)\ s_j$ then $w_i > (=)\ w_j$.*

Proof Suppose $(w_1,...,w_n)$ maximizes the probability of obtaining the tournament T: $L(T,w_1...w_n) = \dfrac{\prod w_i^{s_i}}{\prod_{i<j}(w_i + w_j)}$. Taking the logarithm and putting the partial derivatives equal to zero, we obtain for $i \in \{1,..,n\}$

$$\frac{s_i}{w_i} - \sum_{k\neq i}\frac{1}{w_i + w_k} = 0. \qquad \text{(a)}$$

Hence,

$$s_i = \sum_{k\neq i}\frac{w_i}{w_i + w_k}.$$

If $s_i > s_j$ then

$$\sum_{k \neq i} \frac{w_i}{w_i + w_k} > \sum_{k \neq j} \frac{w_j}{w_j + w_k}.$$

It follows that

$$(w_i - w_j) \sum_{k=1}^{n} \frac{w_k}{(w_i + w_k)(w_j + w_k)} > 0, \text{ thus } w_i > w_j. \blacksquare$$

An iterative scheme for the determination of these weights may be derived from (a) in the proof above, see Moon (1968):

Let $n = |X|$, $w_i^{(o)} = 1/n$ and $w_i^{(k)} = \dfrac{s_i}{\sum_{j \neq i} (w_i^{(k-1)} + w_j^{(k-1)})^{-1}}$ for $i \in \{1,...,n\}$.

Suppose $T \in \mathbf{T}(X)$ is irreducible. Now define: $a \gg b : f_{prob}(T)$ iff $w_a > w_b$ and $a \approx b : f_{prob}(T)$ if $w_a = w_b$. In general, the rule f_{prob} is defined as follows. If $(T,X) = (T,X_1) \gg ... \gg (T,X_n)$, where (T,X_i), $i \in \{1,...,n\}$ is irreducible, then $f_{prob}(T,X) = f_{prob}(T,X_1) \gg ... \gg f_{prob}(T,X_k)$.

COROLLARY 3.24 (Zermelo (1929) and Ford (1957)) *For all tournaments T:*

$$f_{prob}(T) = f_{out}(T). \blacksquare$$

If we try to maximize L(T) under the assumption that $P = (p_{ab})$ satisfies the so-called weak stochastic transitivity condition, we obtain other results:

Thompson and Remage (1964) and Remage and Thompson (1966) let α_{ij} be 1 if alternative a_i wins from a_j, otherwise it is taken to be 0. The likelihood of a tournament T therefore is $L(T) = \prod_{i<j} \pi_{ij}^{\alpha_{ij}} (1-\pi_{ij})^{1-\alpha_{ij}}$. Thompson and Remage propose ranking the alternatives by maximizing L with respect to p_{ij}, subject to the weak stochastic transitivity condition:

Weak stochastic transitivity

A system satisfies weak stochastic transitivity if for all i,j,k:

if $p_{ij} \geq 1/2$ and $p_{jk} \geq 1/2$ then $p_{ik} \geq 1/2$.

Now, suppose that $P = (p_{ij})$ is such a maximizing system, satisfying the weak stochastic transitivity condition. We then may derive (various) linear orders, as follows: $a_1 \gg a_2 \gg ... \gg a_n$ where $p_{ij} \geq 1/2$ if $i < j$.

Thompson and Remage proved that these resulting linear orders are nearest adjoining orders.

The eigenvector method: f_{eigen}

A method due to Wei and Kendall, see Moon (1968) takes into account the quality of the defeated opponents. Suppose that to all players x an initial strength w(x) is assigned. These strengths will be adjusted using the outcomes of the tournament. We may do this by assigning to each player the sum of the initial strengths of the opponents he has beaten. Hence, $\underline{w}_{new} = A\underline{w}_{old}$, where A is the tournament matrix (where $a_{ii} = 0$) and $\underline{w}_{old}$ is the vector of strengths. It would be nice, if we are able to choose the initial strengths $\underline{w}_{old}$, such that $\underline{w}_{new} = \lambda\underline{w}_{old}$ for a real number $\lambda > 0$. For this would mean that the ratios of the ratings given by $\underline{w} = \underline{w}_{old}$ fit the tournament, or are predicted by the tournament. We show that this indeed is possible.

If T is irreducible, the tournament matrix is primitive, which means that there exists an integer n such that A^n has only positive entries. In that case, using a theorem of Frobenius (see for example Bermann and Plemmons 1979), we may deduce that there is a unique positive characteristic root or eigenvalue λ of the tournament matrix A with largest modulus. The eigenspace is 1-dimensional and contains a vector $\underline{w}$ with all components positive. All other eigenvectors of A with all components positive are multiples of $\underline{w}$.

This unique eigenvector $\underline{w}$ is a candidate for the rating: Take $\underline{w}_{old} = \underline{w}$, then $\underline{w}_{new} = A\underline{w}_{old} = \lambda\underline{w}_{old}$, hence all ratios are preserved.

If T is irreducible, we put $a \geq b : f_{eigen}(T)$ iff $\underline{w}_a \geq \underline{w}_b$. If T is reducible, we continue as follows. Let $f_{trans}(T) = X_1 \gg \dots \gg X_k$. Then $f_{eigen}(T) = f_{eigen}(T,X_1) \gg \dots \gg f_{eigen}(T,X_n)$.

These strengths $\underline{w}$ may be determined recursively. First assign to each player the number of matches he has won, giving the vector $\underline{w}_1$, which is equal to $A\underline{e}$, where A is the tournament matrix and $\underline{e}$ is a vector of 1's. Now suppose that $\underline{w}_i$ is given, then take $\underline{w}_{i+1} = A\underline{w}_i$. This means that in going from $\underline{w}_i$, the initial strengths, to $\underline{w}_{i+1}$, each player is assigned the sum of the initial strengths of the players he has beaten. Hence $\underline{w}_i = A^i\underline{e}$. If

(T,X) is irreducible, this process converges to the unique eigenvector, if properly normalized:

$$\lim_{i\to\infty} \left(\frac{A}{\lambda}\right)^i \underline{e} = \lim_{i\to\infty} \frac{w_i}{\lambda^i} = \underline{w}.$$

We illustrate f_{eigen} using the following tournament.

T_1

Then $\underline{w}_1 = (2,2,1,1)$, $\underline{w}_2 = (3,2,1,2)$, $\underline{w}_3 = (3,3,2,3)$, $\underline{w}_4 = (5,5,3,3)$, ... and finally, $\underline{w} = (.827,\ .729,\ .425,\ .593)$ while $\lambda = 1.396$. Note that alternative d receives more weight than alternative c, because d beats the strong alternative a.

If T_2 is as in table 3.1, then f_{eigen} ranks this tournament to $f_{eigen}(T_2) = \{a\} \gg \{b\} \gg \{c\} \gg \{g\} \gg \{d\} \gg \{e\} \gg \{h\} \gg \{f\}$.

We verify some of the properties introduced in previous sections.

First of all, f_{eigen} is not independent of 3-cycle orientation: Take the tournament T_1. Then T_1 is ranked to $\{a\} \gg \{b\} \gg \{d\} \gg \{c\}$. If we reverse the cycle $aT_1cT_1dT_1a$, the ranking becomes $\{a\} \gg \{b\} \gg \{c\} \gg \{d\}$.

The ranking rule f_{eigen} is not weakly Δ-IIA either:

(T_8, X_x):

T_8	a	b	c	d	e	f	g
a	-	1	0	1	1	1	0
b	0	-	1	1	0	1	1
c	1	0	-	0	1	1	1
d	0	0	1	-	0	1	1
e	0	1	0	1	-	1	0
f	0	0	0	0	0	-	1
g	1	0	0	0	1	0	-

$f_{eigen}(T_8,X_X) = \{a\} \gg \{c\} \gg \{b\} \gg \{e\} \gg \{d\} \gg \{g\} \gg \{f\}$
Adding an alternative h between {a,b,c,e} and {d,f,g}, gives:
$f_{eigen}(T_8,X) = \{c\} \gg \{a\} \gg \{b\} \gg \{e\} \gg \{d\} \gg \{g\} \gg \{h\} \gg \{f\}$.

The rule f_{eigen} is not interval consistent. Again take $T = T_1$. Then $f_{eigen}(T_1) = \{a\} \gg \{b\} \gg \{d\} \gg \{c\}$, because alternative d beats the strong alternative a. But $f_{eigen}(T_1,\{c,d\}) = \{c\} \gg \{d\}$.

Concerning the choice set $Max(f_{eigen})$ and the uncovered set, we obtained:

THEOREM 3.25 *For all* $(T,X) \in \mathbf{T}$: $Max(f_{eigen}(T,X)) \subseteq uc(T,X)$.

Proof Suppose xCy in T, where C is the cover relation. Let $\underline{w}$ be the eigenvector, then $A\underline{w} = \lambda\underline{w}$. Hence, $\lambda\underline{w}^u = \sum_{z:\, uTz} \underline{w}^z$ for all $x \in X$. Because xCy, we have $\{z\colon yTz\} \subseteq \{z\colon xTz\}$ and xTy. Thus $\underline{w}^x > \underline{w}^y$. ■

A method due to Wei and Kendall, a long period of time used in chess tournaments, is to replace the score s_i (the number of games won by player i), by the sum of the scores of the players defeated by player i. This new score is equal to $As = A^2\underline{e}$. Note that the eigenvector is a generalization of this idea. Another method is described in Moon (1968), given by Katz and Thompson. They let the vector of relative strength be proportional to $(A + rA^2 + r^2A^3 + \ldots)\underline{e} = A(I-rA)^{-1}\underline{e}$, where r is some positive constant smaller than λ^{-1} (the dominant eigenvalue) so that the series converges. Thompson showed that the relative strengths given by $A(I-\lambda^{-1}A)^{-1}\underline{e}$ are the same as those given by the Kendall-Wei method.

Ratings and tournaments

One may have difficulty with the derivation of ratings from a tournament, which contains (possibly nontransitive) ordinal information only. Nevertheless, we look upon these strengths as being *predicted* or *confirmed* by the tournament, as was the case for f_{prob} and f_{eigen}. We give two other examples. The IQ test consists of a number of small tests for several abilities. Now, some tests are more important than others. Therefore, it is conceivable that one obtains a tournament, where the binary dominance reflects the relative importance of the different tests. The IQ score then, may be seen as a weighted combination of the test scores. In this way the *predictability* of intellectual performance is maximized. See Coombs et al. (1970).

The second example is given by the Elo-ratings used to measure chess skill. Each player's performance, U_i, is assumed to be normally distributed with mean ρ_i (the scale value or 'true rating') and standard deviation σ. Moreover, player i beats player j if $V_{ij} := U_i - U_j > t$, a threshold. Now it is easy to derive that $\text{Prob}(V_{ij} > t) = \Phi[(\rho_i-\rho_j-t)/\sigma] =: f(\rho_i-\rho_j)$. ($\Phi$ is the cumulative distribution function of the standard normal variable.) This is the key assumption used in Elo-ratings: the probability of a win, loss or draw between two chess players only depends on the difference between the two player's ratings. Now, one may use hypothetical scale values $\hat{\rho}_i$, to calculate for each match or tournament the expected number of points, where a win gives two points, a draw one point. It is clear that one may use the tournament results and the points derived from these results, to obtain estimations of $\hat{\rho}_i$. For example, $\hat{\rho}_i = (W_i-L_i)2\sigma/M + \bar{\rho}$, where W_i is the number of wins, L_i the number of losses, M the number of players and $\bar{\rho}$ is the average rating of the opponents, which may arbitrarily be chosen, because ρ is on an interval scale. See Batchelder and Bershad (1979), Batchelder and Simpson (1989) or Elo (1978).

In many situations numerical indices are devised to predict some dependent variable on the basis of some independent variable. In this way we want to look at the use of the eigenvector. If we assume that strength or preference is on a ratio scale we see that the ratings given by this vector are predicted by the tournament.

We end our discussion of ranking rules with a table of some properties.

	interval consistent	separable	Δ-IIA	independent of interval changes	partial independent	independent of 3-cycles
f_{trans}	■	□	■	●	□	■
f_{out}	□	■	□	□	●	■
f^*_{out}	□	■	□	□	□	□
f_{cop}	□	■	□	□	□	□
f_{eigen}	□		□	□	□	□
f_{slat1}	□		□	□	□	□
f_{slat2}	□	■	□	□	□	□
f_{slat3}	■	□	■	□	□	□

□: not satisfied

■: satisfied

● : characterizing properties, if taken together with the elementary commutation properties

Table 3.3 Table of properties.

APPENDIX A.

KERNELS

In this appendix, we restrict our attention to the set $\mathbb{H} \subseteq \mathbb{T}$ of irreducible tournaments. Given the ranking rule $f = f_{out}$, which violates the Δ-IIA condition of section 3.5.3, we want to determine a subset $\Delta(f)$, called a kernel, such that $f|\Delta(f)$ is Δ-IIA.

In this appendix, we write out(x) instead of s_x, the Copeland score of alternative x.

Suppose that we have a set Δ of irreducible tournaments, such that $f|\Delta$ is Δ-IIA. By this we mean the following.

For a ranking rule f on $\mathbb{H}$, $f|\Delta$ is Δ-IIA, iff

for all X, for all $x \in X$, for all $(T,X) \in \mathbb{H}$:

if $(T,X_x) \in \Delta$ and $\forall u \in \text{upp}(x,T)\ \forall w \in \text{low}(x,T)$:$[(u,w) \in a f(T,X_x)]$, then

$f(T,X) = f(T,X_x)|\text{upp}(x,T) \gg \{x\} \gg f(T,X_x)|\text{low}(x,T)$.

We demand one more thing: Δ has to be closed under insertion with respect to f: a subset Δ of $\mathbb{H}$ is called *closed under insertion* with respect to f, iff

for all X, for all $x \in X$, for all $(T,X) \in \mathbb{H}$

if $(T,X_x) \in \Delta$ and for all $u \in \text{upp}(x,T)$ and all $w \in \text{low}(x,T)$:

$[(u,w) \in a f(T,X_x)]$, then

$(T,X) \in \Delta$.

Now we can define:

f-kernel

Let f be a ranking rule on $\mathbb{H}$. A set $\Delta \subseteq \mathbb{H}$ is called an f-kernel iff

(i) *Δ is closed under insertion with respect to f and*

(ii) *$f|\Delta$ is Δ-IIA.*

An f-kernel is written as $\Delta(f)$.

Note that because of the definition of an f-kernel, we may extend $f|\Delta(f)$ to a Δ-IIA ranking rule f^{δ} on $\mathbb{H}$, by defining $f^{\delta}(T,X) = f_{trans}(T,X) = X \times X$ if $(T,X) \notin \Delta(f)$. To show that this is true, we take a tournament $(T,X) \in \mathbb{H}$. Just for convenience, we introduce expression α:

$\alpha = \forall u \in \text{upp}(x,T)\ \forall w \in \text{low}(x,T):[(u,w) \in af(T,X_x)]$.

If $(T,X_x) \in \Delta(f)$ and α, then $(T,X) \in \Delta(f)$, and we know that $f|\Delta(f)$ is Δ-IIA. Now, suppose that $(T,X_x) \notin \Delta(f)$. Then, because we assume that (T,X_x) is irreducible, we know that $f^{\delta}(T,X_x) = f_{trans}(T,X_x) = X_x$. Hence α is not true.

We present two kernels; for f_{out} and for f_{uc}, where uc is the uncovered set. We define $\text{in}(x,(T,X)) = |\{y \in X: yTx \text{ and } y \neq x\}|$.

PROPOSITION 3.26 *We have the following f_{out}-kernel. Take*

$$N := \Delta(f_{out}) =$$
$$\{(T,X) \in \mathbb{H} \mid \text{If } f_{out}(T,X) = Y_1 \gg \ldots \gg Y_p \text{ then } \forall s = 1 \ldots p\ \forall y \in Y_s:$$
$$[out(y,(T,X)) \geq |Y_{s+1}| + \ldots + |Y_p| \text{ and}$$
$$in(y,(T,X)) \geq |Y_1| + \ldots + |Y_{s-1}|]\ \}.$$

Proof This consists of a number of straightforward verifications. ■

Note that this kernel is maximal with respect to set inclusion, in the following sense: if K strictly contains N, then $f_{out}|K$ is not Δ-IIA.

Note that N is not an f^{*}_{out}-kernel: Take the following tournament $(T_9,X_x) \in N$.

T_9	a	b	c	d	e	f
a	-	1	0	1	1	0
b	0	-	1	1	0	1
c	1	0	-	1	1	0
d	0	0	0	-	1	1
e	0	1	0	0	-	1
f	1	0	1	0	0	-

Now $f_{out}(T_9,X_x) = \{a,b,c\} \gg \{d,e,f\}$ and

$f^{*}_{out}(T_9,X_x) = \{a,b,c\} \gg \{d\} \gg \{e\} \gg \{f\}$.

If we add x between e and f we obtain

$f_{out}(T_9,X) = f^*_{out}(T_9,X) = \{a,b,c,d,e,f,x\}$ $((T_9,X) \notin N)$.

PROPOSITION 3.27 *Let K be the subset of* $\mathbb{H}$ *consisting of irreducible tournaments T satisfying the following condition. Consider the rule* f_{uc}, *induced by the choice function uc.*
If $f_{uc}(T) = Y_1 \gg \ldots \gg Y_p$, *then*
for all $s \in \{1,\ldots,p-1\}$ *there is an* $x \in Y_s$ *such that*
for all $u \in \{s+1,\ldots,p\}$ *and all* $y \in Y_u$, *we have:* xTy.
This set K is a (maximal) f_{uc}*-kernel.* ■

Every algorithm f commuting with concatenation has a non-empty f-kernel. Take for example the singletons.

From now on we restrict our attention to the class $\mathbb{O}$ consisting of refinements of f_{out}.

Let M be defined by

$$\begin{aligned}M:= \{(T,X) \in \mathbb{H} : \text{ if } f_{out}(T,X) = Y_1 \gg \ldots \gg Y_n, \text{ then}\\ \forall s\ \forall x \in Y_s\ [\mathrm{out}(x,(T,X)) \geq |Y_{s+1}| + \ldots + |Y_n| - 1\\ \text{and } \mathrm{in}(x,(T,X)) \geq |Y_1| + \ldots + |Y_{s-1}| - 1]\}\end{aligned}$$

The set M may be looked upon as the closure of the set $\Delta(f_{out})$.

THEOREM 3.28 *If* $f \in \mathbb{O}$ *then* $\Delta(f) \subseteq M$.

Proof Let $(T,X) \notin M$. Suppose $f_{out}(T,X) = Y_1 \gg .. \gg Y_n$. Let us assume, because $(T,X) \notin M$, that there is a number s and $y \in Y_s$ such that $\mathrm{out}(y,(T,X)) < |Y_{s+1}| + .. + |Y_n| - 1$. Let $Z = X \cup \{x\}$ and take (T,Z) such that $(T,Z_x) = (T,X)$ and $\mathrm{upp}(x,(T,Z)) = Y_1 \cup \ldots \cup Y_s$ and $\mathrm{low}(x,(T,Z)) = Y_{s+1} \cup \ldots \cup Y_n$. Then, because $(T,X) \in \Delta(f)$ and the fact that $\Delta(f)$ is closed under insertion, we know that $(T,Z) \in \Delta(f)$. Since $(T,Z_x) \in \Delta(f)$ and for all $u \in \mathrm{upp}(x,(T,Z))$ and all $w \in \mathrm{low}(x,(T,Z))$: $u \gg w : f(T,Z_x)$, we may deduce, using the Δ-IIA condition, that $y \gg x : f(T,Z)$.
But, on the other hand, $\mathrm{out}(x,(T,Z)) = |Y_{s+1}| + .. + |Y_n|$, while $\mathrm{out}(y,(T,Z)) < |Y_{s+1}| + .. + |Y_n|$. This implies that $x \gg y : f_{out}(T,Z)$, hence $x \gg y : f(T,Z)$. But this contradicts $y \gg x : f(T,Z)$ which was derived

earlier. ■

If we are looking for possible refinements of the rankings produced by f_{out}, and for matching nontrivial kernels, we have the following three important notes.

Let $(T,X) \in M$ and suppose $f_{out}(T,X) = Y_1 \gg \dots \gg Y_n$.

(i) For all $s \in \{1,..,n\}$ there are $p, q \geq 0$, such that for all $x \in Y_s$

$$\text{out}(x,(T,X)) = (|Y_{s+1}| + \dots + |Y_n| - 1) + p \text{ and}$$
$$\text{in}(x,(T,X)) = (|Y_1| + \dots + |Y_{s-1}| - 1) + q.$$

Because in general $\text{in}(x,(T,X)) + \text{out}(x,(T,X)) = |X|-1$, we have

$$p + q = |Y_s| + 1.$$

(ii) Let $(T,X) \in \Delta(g)$ for an algorithm $g \in \mathcal{O}$. Suppose that g refines Y_s to $Z_a \gg .. \gg Z_b$. When we add an element z between $Y_s \backslash Z_b$ and Z_b we obtain the tournament (T,X'). Now, $\text{out}(z,(T,X')) = |Y_{s+1}| + \dots + |Y_n| + |Z_b|$, and if $u \in Y_s \backslash Z_b$ then $\text{out}(u,(T,X')) = |Y_{s+1}| + \dots + |Y_n| + p$. The rule g^δ being Δ-IIA, this implies that $|Z_b| \leq p$.
Moreover, $\text{in}(z,(T,X')) = |Y_1| + \dots |Y_{s-1}| + |Y_s \backslash Z_b|$, while if $v \in Z_b$, then $\text{in}(v,(T,X')) = |Y_1| + \dots |Y_{s-1}| + q$, implying that $|Y_s \backslash Z_b| \leq q$.
Similarly, we have $|Z_a| \leq q$ and $|Y_s \backslash Z_a| \leq p$.

(iii) If we combine the previous notes, we may deduce that there are at most three possible refinements of an indifference class Y_s of f_{out}:

$Z_a \gg Z_m \gg Z_b$, with $|Z_a| = q-1$, $|Z_m| = 1$ and $|Z_b| = p-1$,
$Z_a \gg Z_b$, with $|Z_a| = q$ and $|Z_b| = p-1$
$Z_a \gg Z_b$, with $|Z_a| = q-1$ and $|Z_b| = p$.

In Monsuur (1990) a class of Δ-IIA algorithms $\{F_{max}\}$ with $\Delta(f_{max}) = M$ for each $f_{max} \in \{F_{max}\}$ was introduced. These algorithms f_{max} are refinements of f_{out} and every strict refinement of an algorithm f_{max} is not Δ-IIA any more.

There is little hope for the existence of a nice $\Delta(f^*_{out})$-kernel. Looking at (i) and (iii) above, we see that possible refinements of an indifference class Y of f_{out} depend on information not available in the tournament

$(T,X|Y)$. On the other hand, the algorithm f^*_{out} only uses information contained in $(T,X|Y)$.

We finally determine subsets of the set of all irreducible tournaments, that are an f-kernel for all f that commute with permutation.

In preparation of proposition 3.29, we introduce the notion of a regular tournament. A tournament (T,X) is regular if for each pair $a,b \in X$, there exists a permutation σ on X, such that $\sigma(a) = b$ and $\sigma(T,X) = (T,X)$. This means that all alternatives play the same role.
Suppose $|X| = m = 2n + 1$. Define (T,X) as follows:
for all $i \in \{1,...,2n+1\}$, $x_i T x_{(i+k)\mathrm{mod}m}$, for $k \in \{1,...,n\}$. It is easy to verify that (T,X) is regular.

PROPOSITION 3.29 *$E = \{(T,X) \in \mathbb{H} : (T,X)$ is regular$\}$ is an f-kernel for all f that commute with permutations.*

Proof Consider an algorithm f that commutes with permutation. Because each alternative plays the same role, $f(T,X) = X \times X$. See proposition 3.1. ■

The union of f-kernels is an f-kernel. A certain f-kernel Δ is said to be maximal if for all subsets B, such that A is strictly contained in B, B is not an f-kernel anymore. If Δ is an f-kernel, then so is $\Delta \cup E$. Hence E is contained in any maximal f-kernel. Conversely, we have:

THEOREM 3.30 *E is the intersection of all maximal f-kernels.*

Proof Let Δ be the intersection. From proposition 3.29, we know that $E \subseteq \Delta$. We show that $\Delta \subseteq E$. Given a tournament (T,X), consider the following equivalence relation $S[T]$ (same role in (T,X)): $aS[T]b$ iff there exists a permutation σ of X such that $\sigma a = b$ and $\sigma(T,X) = (T,X)$. To prove the theorem, it is sufficient to show that if (T,X) is element of the intersection of all maximal f-kernels, there is just one equivalence class of $S[T]$.
To this end, for each choice of tournament (R,X) and each choice of $a \in X$, we introduce the ranking rule $f_{R,a}$. Let A be the $S[R]$-equivalence class of a. Then $f_{R,a}$: $\mathbb{T} \to \mathbb{W}$ is defined by:

$$f_{R,a}(T,Z) = \begin{cases} \sigma(A) \gg X\backslash\sigma(A) & \text{if there is an } \sigma \text{ with } \sigma(R,X) = (T,Z) \\ & \\ ZxZ & \text{else.} \end{cases}$$

It is easy to verify that the ranking produced by $f_{R,a}$ is independent of the choice for σ, therefore it is well-defined.
The rule $f_{R,a}$ commutes with permutation: Take $(T,X) \in \mathbb{H}$ and assume that $\sigma(R,X) = (T,X)$. Now consider $\tau(T,X)$ for a permutation τ of X. Because $\tau\sigma(R,X) = \tau(T,X)$, $f_{R,a}(\tau(T,X)) = \tau\sigma A \gg \tau\sigma(X\backslash A) = \tau[\sigma A \gg \sigma(X\backslash A)] = \tau f_{R,a}(T,X)$.

Now, let (T,X) be a tournament from Δ. Suppose that S[T] has more than just one equivalence class, hence not all elements play the same role in (T,X). We derive a contradiction and are done.
Choose arbitrarily an element $a \in A$ and consider $f_{T,a}$. Because $(T,X) \in \Delta$, we have $(T,X) \in \Delta(f_{T,a})$. Above we have seen that $f_{T,a}(T,X) = A \gg X\backslash A$. Let $Z = X \cup \{x\}$ and (T,Z) is such that $(T,Z_x) = (T,X)$ and $upp(x,(T,Z)) = A$, $low(x,(T,Z)) = X\backslash A$. Then $(T,Z) \in \Delta(f_{T,a})$, because $\Delta(f_{T,a})$ is closed under insertion. Of course there does not exists a permutation σ of X such that $\sigma(T,X) = (T,Z)$, because $|Z| \neq |X|$. Thus $f_{T,a}(T,Z') = Z' \times Z'$ instead of $A \gg \{x\} \gg X\backslash A$, contradicting $f_{T,a}|\Delta(f_{T,a})$ being Δ-IIA. ■

APPENDIX B

INCONSISTENCY IN THE ANALYTIC HIERARCHY PROCESS

AND

THE INSERTION OF NEW ALTERNATIVES

1 Introduction

One of the important aspects of the Analytic Hierarchy Process (AHP) is the allowance for inconsistency in the pairwise comparisons. It is the presence of inconsistency which necessitates using elaborate estimation methods. Only when a matrix is consistent, the computation of relative weights reduces to a simple normalization of one of its columns. Due to the fact that many applications have been made of the AHP for decisions in complex situations, several methods for computation of the relative weights have been advocated. One of them is the eigenvalue technique, developed by Saaty (1980), which will be introduced in section 2. This method assumes that inconsistencies are due to the error in preference measurement. In a process of minimizing these errors, one therefore may utilize a measure of inconsistency.

Most of the other techniques have a statistical nature. In these approaches, inconsistency is inherent in the preferences. For example, often it is assumed that the input data always have errors in the form of

$$a_{ij} = \frac{w_i}{w_j} \varepsilon_{ij},$$

where the error ε_{ij} has a certain distribution. The most important problem then is the identification of the statistical model that can best explain the variability in the data (see Dadkhah and Zahedi, 1993). Developing inconsistency measures is meaningless, because there is no justification to the assertion that a consistent matrix is less random than an inconsistent matrix.

In this appendix, we follow the approach of Saaty. We assume that inconsistencies may arise if for example one is not trained to determine ratios or because the precise criteria that have to be used in the evaluation of the alternatives are not yet crystal-clear. In any case, we think that an inconsistency measure can help the decision maker to move towards his true preferences. It is one of the advantages of the AHP that it is equipped with an inconsistency measure. In section 3, we show a connection between this inconsistency measure and the phenomenon that new alternatives that are included consistently in the analysis of a decision problem, systematically loose (relative) weight. Finally, in section 4, we suggest an alternative consistency check.

2 The eigenvalue technique and inconsistency measure of Saaty

Let X be the set of alternatives and let $|X| = n$. We consider a reciprocal matrix $R = (r_{ij})$. A matrix R is reciprocal if $r_{ij}r_{ji} = 1$ for all choices $i,j \in \{1,...,n\}$. For each i,j, this r_{ij} is the estimated ratio of weights or strengths w_i and w_j of alternative i respectively j. Given the reciprocal matrix, we want to use an estimation technique, producing a vector of weights $\underline{w}$, such that, in general, r_{ij} is approximately equal to the calculated $\underline{w}_i/\underline{w}_j$. This rating is unique up to a multiplication with a positive constant α. If the matrix R is consistent, a *natural* rating of the alternatives is given by an arbitrary column of R. If R is *inconsistent*, the columns may yield different ratings.

The eigenvalue technique, f_{ev}, orders the alternatives on a ratio scale, according to the components of the eigenvector $\underline{w}$ that corresponds to the unique eigenvalue λ_{max} with largest modulus. This eigenvector may be obtained by using the power method:

$$\underline{w} = \lim_{t \to \infty} \left(\frac{R}{\lambda_{max}}\right)^t \underline{e} \text{ ,}$$

$\underline{e}$ being a column vector of 1's.

To give an example, consider the following 3x3 reciprocal matrix.

$$R = \begin{pmatrix} 1 & 2 & 5 \\ 1/2 & 1 & 7 \\ 1/5 & 1/7 & 1 \end{pmatrix}$$

Then $f_{ev}(R) = (0.5415, 0.3816, 0.0768)$, while $\lambda_{max} = 3.1185$. In general, $\lambda_{max} \geq n$ and $\lambda_{max} = n$ iff R is consistent.

The inconsistency measure used by Saaty is $\mu = (\lambda_{max}-n)/(n-1) \geq 0$, which is the mean of the other eigenvalues. It is equal to

$$-1 + \frac{1}{n(n-1)} \sum_{1\leq i<j\leq n} (\varepsilon_{ij} + \frac{1}{\varepsilon_{ij}}),$$

where $\varepsilon_{ij} = r_{ij}w_j/w_i$. See Saaty (1980). If $\varepsilon_{ij} \neq 1$, then there is an inconsistency. This formula shows that μ is a convex function of ε_{ij}. The function μ attains its minimum value when $\varepsilon_{ij} = 1$.
We show that 2μ is the average of the (local) inconsistencies given by $(\varepsilon_{ij} + \frac{1}{\varepsilon_{ij}} - 2)$.

$$\begin{aligned}\frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} (\varepsilon_{ij} + \frac{1}{\varepsilon_{ij}} - 2) &= -2 + \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} (\varepsilon_{ij} + \frac{1}{\varepsilon_{ij}}) \\ &= -2 + 2(\mu + 1) \\ &= 2\mu.\end{aligned}$$

To establish a consistency threshold, compare μ to the average value $\bar{\mu}$ of randomly generated matrices of the same dimension. If $\mu/\bar{\mu}$ is too large, one would recommend that the decision maker revise his elicited preferences.

3 Saaty's measure and the addition of alternatives

Suppose that after applying the eigenvalue technique f_{ev} to a reciprocal matrix R, we obtain a vector of weights $\underline{w} = (w_1,\dots,w_n)$. Of course, one might as well take $\alpha\underline{w}$. Taking a certain value for α, may be interpreted as fixing the unit of the ratio scale.

After choosing α, we consider a new alternative. If we compare this new alternative with the first alternative, it gives a certain ratio, say ρ. Hence we take as the value of this new alternative $c = \rho w_1$. Therefore, one might suggest that the rating of these n+1 alternatives should be equal to $(w_1,\dots,w_n,\rho w_1)$. But the eigenvalue technique provides a different rating, as will be shown in theorem 3.31.

To introduce this theorem, we compare the new alternative with every other alternative from $X = \{x_1,...,x_n\}$ and assume that this happens in a consistent way. Since the weight of the new alternative x_{n+1} is supposed to be equal to $c = \rho w_1$, we might construct a reciprocal matrix as follows. To the original matrix R we add a row and column. Further, take $r_{mm} = 1$, $r_{im} = w_i/\rho w_1$ and $r_{mi} = \rho w_1/w_i$, $m = n+1$ and $i \in \{1,...,n\}$. The resulting reciprocal matrix is denote by R'.

To give an example, take the reciprocal matrix of the previous section. Let c, the weight of the fourth alternative be equal to 0.55. Then we obtain the matrix

$$R' = \begin{pmatrix} 1 & 2 & 5 & 0.984 \\ 1/2 & 1 & 7 & 0.693 \\ 1/5 & 1/7 & 1 & 0.139 \\ 1.015 & 1.441 & 7.161 & 1 \end{pmatrix}$$

Now $f_{ev}(R') = (0.5415, 0.3816, 0.0768, 0.534)$. Observe that the weight of the new alternative, originally 0.55, has become 0.534. The other weights remain as they were before. This is no accident, as the following theorem shows.

THEOREM 3.31 *If $f_{ev}(R) = \underline{w}$, then $f_{ev}(R') = \underline{w}' = (\underline{w}, kc)$,*

$$\textit{where } k = \frac{1 - \lambda_{max} + \sqrt{(\lambda_{max} - 1)^2 + 4n}}{2} > 0.$$

Proof Let $\underline{w}' = (\underline{w}, kc)$. Then $R'\underline{w}' = ((\lambda_{max}+k)\underline{w}, (n+k)c)$. If we compute k such that $\lambda_{max} + k = (n + k)/k$, we obtain the k given above. Hence $R'\underline{w}' = (n+k)/k\ \underline{w}'$. Because $k > 0$, $\underline{w}' > 0$. Therefore, using a theorem of Perron-Frobenius (see for example Bermond and Plemmons, 1979), this $\underline{w}'$ is the eigenvector corresponding to the unique eigenvalue with modulus equal to the spectral radius. ∎

In the example above, k is equal to 0.971. In fact, the number k is always smaller than or equal to 1:

PROPOSITION 3.32 *The number k in theorem 1 has the following properties. First of all, $0 < k \leq 1$ and $k = 1$ iff R is consistent. Furthermore, k is a strictly descending function of λ_{max} (and μ).*

Proof As shown in theorem 3.31, if λ_{max}' is the eigenvalue with largest modulus of R', then $\lambda_{max}' = (n+k)/k$. Since, as is shown by Saaty (1980), $\lambda_{max}' \geq n+1$, we obtain $n + k \geq kn + k$, which gives $n \geq kn$. We obtain $k \leq 1$. Above we showed that $0 < k$. It is easy to prove that $k = 1$ iff $\lambda_{max} = n$, which is equivalent with R being consistent. Differentiating the function $k(\lambda_{max})$, we obtain that $k'(\lambda_{max}) < 0$. ∎

Theorem 3.31 and proposition 3.32 imply that even if $c > w_1$, it may happen that $kc < w_1$, so we have a kind of rank reversal. In any case, a newly added alternative is always somewhat at a disadvantage. Moreover, as is shown above, the cause of this kind of violations may be ascribes to the inconsistency of the reciprocal matrix R.

Finally, it is straightforward to prove:

PROPOSITION 3.33 *The inconsistency $\mu(R')$ of R' is less than the inconsistency $\mu(R)$ of R: $\frac{1-k}{k} < \frac{(n+k)(1-k)}{k(n-1)}$. Moreover, $k(R') > k(R)$.* ∎

4 An alternative consistency check?

To introduce a possibly new consistency check, let s_i be the sum of the elements of row i: $s_i = \sum_j r_{ij}$. We have the following conjecture:

CONJECTURE *The following three statements are equivalent.*

(i) *R is consistent*

(ii) $\sum_{i=1}^{n} \frac{1}{s_i} = 1$

(iii) *The harmonic mean of the row sums s_i is equal to n.*

The proof of the equivalence of (ii) and (iii) is straightforward.

For arbitrary R let us consider the rating rule, which assigns to R the rating given by the row sums s_i. In the same manner as is described in section 3, we may add a new alternative in a consistent way with weight $c = \rho s_1$. It is straightforward to prove that if we apply the fore mentioned rating rule to R', we obtain the rating

$$(s_1, \ldots, s_n, \quad \frac{1 + c\sum_{i=1}^{n} \frac{1}{s_i}}{1 + c} \quad c),$$

which completes the proof of (i) $\Rightarrow$ (ii).

The converse in case $n = 3$ can be proved by elementary means. The proof for $n > 3$ remains open.

CHAPTER 4

SOLUTIONS FOR DOMINANCE STRUCTURES

4.1 INTRODUCTION

Perhaps the best known chessboard covering problem is the determination of the minimum number of queens required to cover the entire chessboard. A configuration of queens on an m×m chessboard is said to dominate the board if every square either contains a queen or is attacked by a queen. The configuration is said to be non-attacking if no queen attacks another queen. The best upper bound known for the minimum number f(m) of non-attacking queens needed to dominate an m×m chessboard is
$f(m) \leq \frac{14}{23}m + O(1)$. See Grinstead et al. (1991).

For example, on a standard chessboard, we need five queens, see figure 4.1.

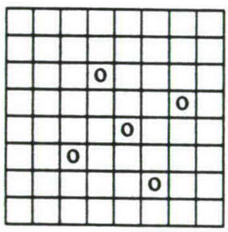


Figure 4.1 A minimal non-attacking configuration, see Cockayne (1991).

In this chapter, we consider asymmetric relations. These relations may for instance, represent direct domination between parties or countries. This direct domination means that one party can bend the other one to its will. Or, when parties are sitting down at the negotiation table, it means that the stronger party can set the conditions under which cooperation or trade will take place, compare van den Brink and Gilles (1992).
Of course, in such relations, which we represent graphically using a directed graph, we may look for dominating sets, as was done in the queens problem.

A dominating set that closely corresponds to the queens problem, is the one introduced by von Neumann and Morgenstern (1953, section 65), which we shall discuss hereafter. Of course, the queens problem is different; it concerns a bi-directed graph. The vNM-concept has found many applications, for instance in cooperative game theory, see Owen (1982), but also in social sciences, see van Deemen (1991), or von Neumann and Morgenstern (1953, page 41,42).

A subset Y of a set X is a von Neumann-Morgenstern stable set (vNM-stable set) of an asymmetric relation if there is no domination inside Y, and for all alternatives outside Y there is at least one alternative in Y that dominates it. So it is free of internal conflicts and dominates the rest of the alternatives. More precisely:

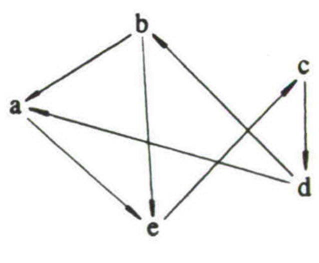
Definition 4.1

A set $Y \subseteq X$ is a von Neumann-Morgenstern stable set of the asymmetric relation (G,X) if

(a) for all $a,b \in Y$: $\neg\, aGb$, *[internal stability]*

(b) for all $a \notin Y$ there is an $b \in Y$ such that bGa. *[external stability]*

To give an example, we consider the relation (G_1,X) of figure 4.2:



vNM-stable sets: {b,c}, {d,e}.

Figure 4.2 A relation (G_1,X) with two vNM-stable sets.

The general idea is that only these vNM-stable sets will be formed in the process of arriving at an enduring and decisive coalition: it is free of internal conflicts and it dominates all other parties. Note that this final domination may be quite different from the direct domination via the relation G, as is illustrated by the asymmetric relation of figure 4.2. There we have a vNM-stable set equal to {d,e}, while aG_1e. Hence this direct domination is overruled by the global domination, expressed by the vNM-stable set.

Again, we let $U = \{u_1,...,u_n\}$ be a finite set of alternatives. The set $\mathbf{G}(X)$, $\emptyset \neq X \subseteq U$, is the set of asymmetric relations on X, while $\mathbf{G}$ is the union of all sets $\mathbf{G}(X)$. In section 4.2, we consider acyclic relations:

We let $\mathbf{A} \subseteq \mathbf{G}$ be the set of acyclic relations. For $\emptyset \neq X \subseteq U$, we let $\mathbf{A}(X)$ be the set of acyclic asymmetric relations on X.

4.2 DOMINATION ON ACYCLIC ASYMMETRIC RELATIONS

In this section, we concentrate on acyclic asymmetric relations. Besides the von Neumann-Morgenstern concept leading to the notion of a dominating coalition, we introduce a few other solution concepts. After that, we state two characterizations of the unique von Neumann-Morgenstern stable set. The first one is in terms of necessary and sufficient conditions for a choice function, for the second one we use zero-sum game theory. Moreover, we introduce a power index for the members of this unique stable set.

4.2.1 Introduction of some choice functions

We discuss four different solution concepts. It will turn out that two of these lead to the same coalition whenever the relation is acyclic.

We start with the vNM-stable sets.

von Neumann-Morgenstern: vNM

In many situations an asymmetric dominance relation G is acyclic, e.g. in hierarchies. In such circumstances it is conceivable that one tries to find a subset or group Y of the set of alternatives X that dominates all other alternatives.

As already mentioned in the introduction, a well-known dominating coalition is given by the vNM-stable sets. In case of acyclic asymmetric relations, we have the following theorem, see von Neumann-Morgenstern, 1953, section 65.8.

THEOREM 4.1 *All acyclic asymmetric relations G on finite X do have a unique vNM-stable set.*

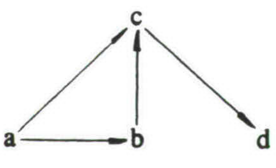
Proof We use induction to $|X|$. If $|X| = 1$, the proof is obvious. Now, suppose that the theorem is true for X such that $|X| \leq k$, $k \in \mathbb{N}$. Let (G,X) be an acyclic asymmetric relation on X, $|X| = k + 1$.
The relation (G,X) is acyclic, so $\mathrm{Max}(G,X) \neq \emptyset$. Because a vNM-stable set N is externally stable, $M = \mathrm{Max}(G,X) \subseteq N$. Since N is internally stable, the set of elements of D, where $D = \{x \in X\text{: there is an } y \in M \text{ such that } yGx\}$, is contained in $X \setminus N$. Let $Y = X \setminus (M \cup D)$. Because $G|(M \times Y \cup Y \times M) = \emptyset$, a vNM-stable set N is the union of M and a vNM-stable set of (G,Y). Using the induction hypothesis, we know that (G,Y) has a unique vNM-stable set, which completes the proof. ∎

If X is infinite, theorem 4.1 is not true; the existence nor the unicity is guaranteed.

We now are able to define the choice function $vNM : \mathbb{A}(X) \to 2^X \setminus \{\emptyset\}$.

For all $G \in \mathbb{A}(X)$, $vNM(G)$ *is equal to the unique vNM-stable set.*

To give an example, consider the following acyclic asymmetric relation (G_2,X).



Then $vNM(G_2,X) = \{a,d\}$.

Besides the notion of vNM-stable sets, we consider three other concepts: top elements, uncovered elements and the strategy equilibrium.

Maximal or top elements: top

Let (G,X) be an acyclic asymmetric relation on X. Then the choice function *top* is defined by:

$$top(G) = Max(G).$$

In the example above, *top*(G_2) = {a}.

Uncovered elements: uc

As defined in chapter 3, for a relation (G,X) ∈ $\mathcal{G}$(X), *uc*(G,X) = Max(C,X), where C is the cover relation.

In the example above, *uc*(G_2) = {a,c}.

The strategy equilibrium: strat

In chapter 2, we introduced the choice function *strat*. As in the case of tournaments, we may search for optimal strategies or mixed strategy equilibria. Unfortunately, these need not be unique for arbitrary asymmetric relations, contrary to the case of tournaments. To give an example, consider once more the acyclic asymmetric relation (G_2,X).

In that case $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. The set of symmetric optimal strategies consists of

$\underline{p} = \lambda(1,0,0,0) + (1-\lambda)(1/2,0,0,1/2)$, where $\lambda \in [0,1]$.

In case G is a tournament, we know from Laffond et al. (1993) and Fisher and Ryan (1992), that the number of alternatives in the support of an optimal strategy is odd. In case of asymmetric relations, this is not always true, as the foregoing example shows.

In the example above, *strat*(G_2) = {a,d}.

In theorem 4.2 we prove that in case of acyclic asymmetric relations G, the union of supports of the optimal strategies, *strat*(G) is precisely the unique vNM-stable set of G.

THEOREM 4.2 *For all acyclic asymmetric relations G, we have that* $vNM(G) = strat(G)$.

Proof We prove that *strat*(G) is the unique vNM-stable set. Therefore, we first prove that *strat*(G) is externally stable.
Note that for arbitrary strategies $\underline{p}$ and elements x,

$$(\underline{p}A)_x = \sum_{y\,:\,yGx} \underline{p}_y - \sum_{y\,:\,xGy} \underline{p}_y.$$

Since, if $z \notin strat(G)$, $\underline{p}_z = 0$ for all optimal strategies, we may deduce that there is an optimal strategy $\underline{q}$, such that $(\underline{q}A)_z > 0$, meaning that there is an $y \in strat(G)$ such that yGz.
As for the internal stability: suppose that $x,y \in strat(G)$ and xGy. Then, to assure that for an optimal strategy $\underline{p}$ such that $\underline{p}_x > 0$, we have $(\underline{p}A)_y = 0$, there must be an $z \in supp(\underline{p}) \subseteq strat(G)$, such that yGz. If we pursue this line of reasoning, we obtain an infinite sequence of alternatives $x_0, x_1, \ldots$ such that for all $i \geq 0$, $x_i G x_{i+1}$. But, since G is acyclic, this means that all x_i are different, implying that X is infinite, which is not the case. ■

For another proof of theorem 4.2, we refer to theorem 4.13.
Concerning the (non-) unicity of optimal strategies, we present

THEOREM 4.3 *Let* $(G,X) \in \mathbb{G}(X)$ *be acyclic. Then there is a unique optimal strategy iff* $Best(G,X) \neq \emptyset$.

Proof It is easy to see that in case $|Max(G,X)| > 1$ or in case $Max(G,X) = \{x\}$ and there is an $y \in X$ such that $\neg xGy$, we have several optimal strategies. Conversely, if the unique vNM-stable set is equal to $\{x\} = Best(G,X)$, then there is a unique optimal strategy $\underline{p}$ given by $\underline{p}_y = 0$ if $y \neq x$ and $\underline{p}_x = 1$. The strategy $\underline{p}$ is unique, because $(\underline{p}A)_y > 0$ for all $y \neq x$, implying that for all other optimal strategies $\underline{q}$ and all $y \neq x$, we have $\underline{q}_y = 0$. ■

In the case of tournaments, we have: $strat(T,X) \subseteq uc(T,X)$, see chapter 2, table 2.2. For arbitrary acyclic asymmetric relation, this does not hold. Consider the example given in the introduction of the choice function *vNM*. Then $strat(G_2) = \{a,d\}$, while $uc(G_2) = \{a,c\}$.

4.2.2 Characterizations; the acyclic case

In this section we provide two characterizations of the choice function *strat* on $\mathbb{A}$, the acyclic asymmetric relations. As is proved in theorem 4.2, on acyclic asymmetric relations, *vNM* = *strat*, so the same characterizations apply to *vNM* as well.

In the characterization that will be presented in theorem 4.5, we use the notion of tail stability. This was first introduced in chapter 2 section 2.7.2. We repeat the definition.

- **Tail stability**

  A choice function S on $\mathbb{A}$ *is tail stable, iff*
  for all X and all $(G,X), (G',X) \in \mathbb{A}(X)$,
  if $G \cap [X^2 - (X - S(G,X))^2] = G' \cap [X^2 - (X - S(G,X))^2]$ *then*
  $S(G,X) = S(G',X)$.

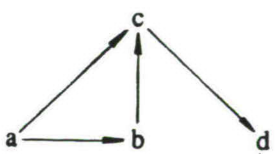
The asymmetric relation (G,X) may be replaced by another acyclic asymmetric relation (G',X) without changing the choice set, if the differences between (G,X) and (G',X) are limited to comparisons between elements of $X \backslash S(G,X)$.

Not every choice function is tail stable. For example $S = top$ is not tail stable, as is shown in the following example $(G_3, \{a,b,c\})$.

$a \longrightarrow b \longrightarrow c \qquad top(G_3) = \{a\}$

$a \longrightarrow b \longleftarrow c \qquad top(G'_3) = \{a,c\} \neq top(G_3).$

Also the choice function *uc* is not tail stable, as is shown using once more the relation G_2:



In this example, $uc(G_2) = \{a,c\}$, but if we add bG'_2d, then c becomes covered in G'_2.

In the introduction of this chapter, we mentioned domination in terms of setting the conditions under which cooperation will take place. In that case, if the parties are sitting down at the negotiation table, it is possible that party *a*, compared to party *b*, knows itself to be supported by a majority of external voters. Indeed, in most cases, the dominance relation (G,X) is settled before the negotiations start. Then, the assumption of tail stability means that the coalition S(G,X) can withdraw from 'the scene of battle', paying attention to more important business. They are not bothered with fights between losers in X\\$S$(G,X) in the tail of the ranking $S(G,X) \gg X \backslash S(G,X)$. No matter how the comparisons between losers change, the choice set remains the same. Moreover, for a party not in S(G,X), in order to become one of the champions, it has to focus on the comparison with parties from the choice set. In our example, it has to present better proposals to the external voters.

LEMMA 4.4 *The choice function vNM ≡ strat on* $\mathbb{A}$ *is tail stable.*

Proof As is proved in theorem 4.1, there is just one vNM-stable set N. Suppose that we replace an acyclic relation (G,X) by another acyclic one, let us say (G',X), such that the relationship between (G,X) and (G',X) is as described in the definition of tail stability. Then N is the unique vNM-stable set of G'. ■

A second property that we need in theorem 4.5, is the Condorcet principle:

- **Condorcet principle**

  Let S be a choice function on $\mathbb{A}$. *Then S satisfies the Condorcet principle, if for all* $(G,X) \in \mathbb{A}$, *we have* $Max(G,X) \subseteq S(G,X)$.

If an alternative is unbeaten in $(G,X) \in \mathbb{A}(X)$, we want it to be member of the dominating coalition.

Finally, we describe the behavior of the choice function S with respect to local differences between two alternatives a and b.

- **Local differences**

  A choice function S respects local differences, if for all acyclic asymmetric relations (G,X) and all $a,b \in X$, the following holds.

  If aGb then $S(G,X) \cap \{a,b\} \neq \{a,b\}$.

If aGb, then this condition implies that it is not allowed that both alternatives are in the choice set. So the difference between a and b is not smoothed out. Note that this condition does not exclude the possibility that alternative b is chosen instead of a, even though aGb. This phenomenon clearly illustrates that we have two levels of analysis. First of all, we have a local level of direct, binary domination, which expresses the ability of an alternative to bend others to its will. On the other hand, we have a more global level, where the strength of an alternative lies in its nesting among a dominating coalition.

Now we are in a position to state and prove a characterization of the choice function $S = vNM = strat$ on $\mathbb{A}$.

THEOREM 4.5 (Characterization of *vNM* and *strat* on $\mathbb{A}$) *Let S be a choice function on $\mathbb{A}$.*

Then $S = vNM = strat$ iff

- *S is tail stable*
- *S satisfies the Condorcet principle*
- *S respects local differences.*

Proof (only if). The tail stability is provided by lemma 4.4. Using the external stability, we may prove that *vNM* satisfies the Condorcet principle. Finally, the internal stability of the unique vNM-stable set implies that *vNM* does respect local differences.

(if). We prove that S(G) satisfies the internal and external stability, described in definition 4.1. The internal stability is equivalent to S respecting local differences. We prove the external stability. Suppose $y \notin S(G)$ and for all $x \in S(G)$, we have that $\neg xGy$. By deleting the comparison zGy for each $z \notin S(G)$, we obtain $G' \in \mathbb{A}(X)$, such that $y \in Max(G')$. Since S satisfies the Condorcet principle, this means that $y \in S(G')$. Further, because S is tail stable, we have that $S(G) = S(G')$, which contains the alternative y. But this contradicts $y \notin S(G)$. ∎

The logical independence of the conditions:

(*A*) For all (G,X), let *S*(G,X) = X. Then *S* satisfies the first two conditions; the third is violated.

(*B*) Define *S*(G,X) := *vNM*(vG,X), where (vG,X) = $\{(x,y) \in X^2: yGx\}$. This rule satisfies conditions 1 and 3, but clearly violates condition 2.

(*C*) The choice function *top* satisfies condition 2 and 3. Condition 1 is violated as is shown using the relation (G_3,X).

REMARK Suppose that we replace (G,X*vNM*(G,X)) by an acyclic relation (G',X*vNM*(G,X)). Then it is possible that (G',X) is cyclic: Let (G,X) be the relation pictured below.

Here, *vNM*(G,X) = {a,b}. If we change the outcome between alternatives c,d $\in$ X*vNM*(G,X) to cG'd, we obtain a cyclic relation.

But, of course, for cyclic relations too, we may compute vNM-stable sets. It is easy to see that also in this case, {a,b} is the unique vNM-stable set.

In the second characterization, we use game-theoretic arguments, which will provide a power index for the alternatives of *vNM*(G,X).

One problem for a newly established coalition N may be the determination of a satisfactory division of power. By this we mean the following. For instance, in the relation G_2, we have the vNM-stable set N = {a,d}. The alternative c is dominated by N through the alternative a. Now, a satisfactory distribution of power may be a vector $(\underline{p}_a,\underline{p}_d)$, such that $\underline{p}_a > \underline{p}_d$. In that case, for the alternative c, the sum of weights of elements from N that it dominates, is smaller than the sum of weights of elements of N by which it is dominated.

In general, we search for a division, such that

$$\text{for all } x \notin N: \sum_{y \in N,\, yGx} \underline{p}_y > \sum_{y \in N,\, xGy} \underline{p}_y.$$

Theorem 4.6 shows that there is just one such coalition:

THEOREM 4.6 (Characterization of *strat* = *vNM*) *Let (G,X) be an acyclic asymmetric relation.*

Then N := strat(G,X) = vNM(G,X) is the unique subset of X

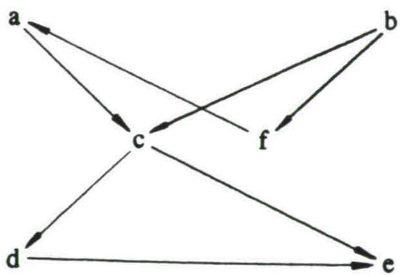
(I) that is internally stable and

(II) that is support of a distribution of weights $\underline{p} \geq \underline{0}$ *summing to 1, satisfying: if* $x \notin N$ *then* $\sum_{y \in N,\, yGx} \underline{p}_y > \sum_{y \in N,\, xGy} \underline{p}_y$.
(strong external stability)

Proof Suppose that N is internal stable and allows a distribution of weights as described in (II). Then is is clear that N is externally stable. Hence N is the unique vNM-stable set. Conversely, if we take N = *strat*(G,X). we may construct an optimal strategy $\underline{p}$ with support *strat*(G,X), such that for all $x \notin N$, we have that $(\underline{p}A)_x > 0$, which is equivalent with (II). ■

This theorem may be seen as giving a more refined meaning to the notion of external stability. The weights $\underline{p}_x$ can be viewed as power indices.

To give an example, consider the following acyclic asymmetric relation (G_4,X):



Then the unique vNM-stable set is equal to {a,b,d}. Now, possible divisions of power between a,b and d, satisfying the conditions of the theorem above, are the ones satisfying $\underline{p}_a + \underline{p}_b > \underline{p}_d$ (domination of c), $\underline{p}_a < \underline{p}_b$ (domination of f), $\underline{p} > \underline{0}$ and $\underline{p}_a + \underline{p}_b + \underline{p}_d = 1$.
These are $\lambda_1(0,1,0) + \lambda_2(1/2,1/2,0) + \lambda_3(0,1/2,1/2,) + \lambda_4(1/4,1/4,1/2)$, where $\Sigma^4_{i=1}\lambda_i = 1$, $0 < \lambda_i < 1$, $i \in \{1,...,4\}$. See figure 4.3.

Let K^n, $n \in \mathbb{N}$, be $\{\underline{p} \in \mathbb{R}^n\colon \underline{p} \geq \underline{0}$ and $\Sigma \underline{p}_i = 1\}$, the set of mixed strategies.

Take (G,X) $\in$ $\mathbb{A}$(X). Let rel(O(G)) be those vectors in K^m, m = $|$*strat*(G,X)$|$, that are restrictions of an optimal strategy $\underline{p}$ $\in$ O(G) to *strat*(G,X). Hence, we delete (zero-) components outside *strat*(G,X). We give rel(O(G)) the induced topology from $\mathbb{R}^m$. Then we have:

PROPOSITION 4.7 *Take (G,X) $\in$ $\mathbb{G}$(X) to be acyclic. Then $\underline{p}$ satisfies the conditions of theorem 4.6, iff*
[O(G) = {$\underline{p}$} or $\underline{p}$ $\in$ int(rel(O(G))]. ■

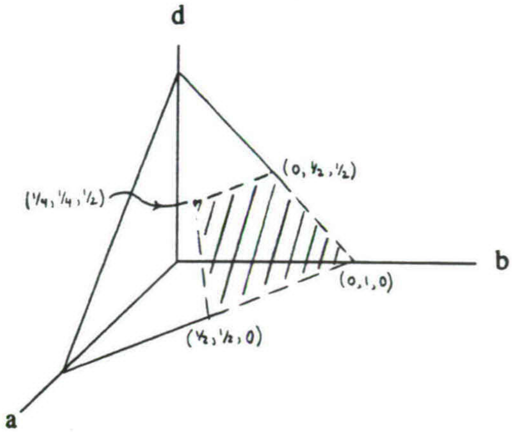


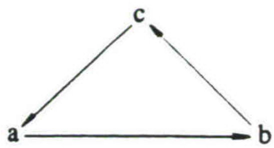
Figure 4.3 Possible divisions of power between a,b and d.

4.3 GENERALIZED vNM-STABLE SETS

In this section, we introduce and discuss two modifications of von Neumann-Morgenstern stable sets.

4.3.1 The drawback of vNM-stable sets

As we have seen in theorem 4.1, in case of acyclic asymmetric relations, there is just one vNM-stable set. In general this is not the case. As is shown in figure 4.2, there may be several vNM-stable sets. Moreover, it also may happen that there is none. Take a circular relation on three alternatives as follows:



All subsets of X = {a,b,c} either violate the internal or the external stability.

4.3.2 Generalized stable sets

Because of the drawback illustrated above, van Deemen (1991) introduced generalized stable sets.

Instead of direct domination of the alternatives via G, he considers domination via the transitive closure τG.

Definition 4.2 *(van Deemen, 1991)*
A nonempty subset V of X is a generalized stable set of (G,X), if

(a) for all $x \neq y \in V$: $\neg x(\tau G)y$,

(b) for every $y \in X \backslash V$ there is an $x \in V$ such that $x(\tau G)y$.

Van Deemen (1991, page 114) showed that for each asymmetric relation, there exists a generalized stable set. In our example above, there are three such sets: {a}, {b} and {c}, while in the relation of figure 4.1, there are 5 generalized stable sets: {a}, {b}, {c}, {d} and {e}. In van Deemen (1991), the emphasis lies on the internal stability.

Unfortunately, a vNM-stable set is not necessarily a generalized stable set. Indeed, if we again take the relation of figure 4.2 given in the introduction, we see that the vNM-stable sets {b,c} and {d,e} are no generalized stable sets, since they do not satisfy condition (a).

This led us to introduce generalized vNM-stable sets.

4.3.3 Generalized vNM-stable sets

We continue with the definition of generalized vNM-stable sets. Subsequently, we will discuss its merits. Let $a\tau(G,V)$ be the asymmetric part of the transitive closure of (G,V), which is the restriction of G to

V.

Definition 4.3

A nonempty subset V of $X \subseteq U$ is a generalized vNM-stable set of $G \in \mathbb{G}(X)$, if

(a) *for all $x,y \in V$: $\neg\, x[a\tau(G,V)]y$,*

[generalized internal stability]

(b) *for every $y \in X \setminus V$ there is an $x \in V$ such that xGy.*

[external stability]

The concept of generalized vNM-stable set is an extension of the vNM-stable sets:

PROPOSITION 4.8 *Let $G \in \mathbb{G}(X)$. If V is a vNM-stable set of G, it is a generalized vNM-stable set of G.* ■

From its definition it is clear that, compared with the original vNM-stable sets, we have changed the notion of internal stability, while keeping the external stability. We may give several reasons for these choices.

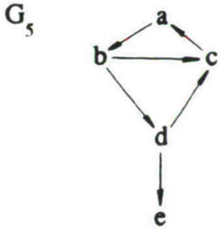
First, recall that we search for dominating sets, such that they always exist. Defining generalized internal stability and external stability as above, assures existence of the generalized vNM-stable sets (see the construction theorem 4.9 hereafter).

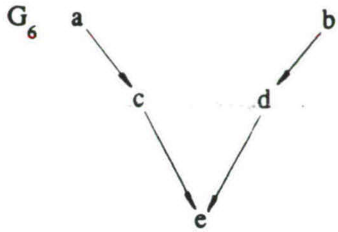
Secondly, remembering the queens problem, we see that it is important that we can reach all squares in just one move. This may also be important in other situations. We want a dominating coalition to be determined and decisive. Therefore, we prefer the direct domination, because often external communication is expensive, laborious and not very reliable. We do not want to depend on an intermediary.

Thirdly, the generalized internal stability as given in definition 4.3, means that for all d,t $\in$ V, if there is a path in V along G from d to t, then there is a path from t to d in V, which means that inside V the domination via τG is symmetric. In other words, if a dominates b in V, then it lacks backing inside V. If a can set the conditions under which cooperation will take place, b can do the same, but then indirectly, via elements of V. It represents a balanced situation.

Finally, it may be desirable for a coalition that it can put pressure on a partner. For example, when the coalition has to go into action or when they want the partner to join an operation.

In figure 4.4 we give two examples of generalized vNM-stable sets.





{a,d}, {a,b,c,d}, {a,b,c,e}

{a,b,e}

Figure 4.4. Illustration of generalized vNM-stable sets.

The following theorem shows how to construct all generalized vNM-stable set. Moreover, it shows that there always is at least one generalized vNM-stable set.

$\mathrm{Max}(\tau(G,X)) = \{x \in X$: there is no y such that $y[a\tau(G,X)]x\}$, where $a\tau(G,X)$ is the asymmetric part of the transitive closure.

CONSTRUCTION THEOREM 4.9 *Every generalized vNM-stable set S of an asymmetric relation (G,X) can recursively be (re-) constructed in the following way.*

Step 1.

Choose a generalized vNM-stable set S_0 for $(G,Max(\tau(G,X))$.
Let D_0 be the set of all alternatives of $X \backslash S_0$ that are dominated via G by at least one element of S_0. Next $Y := X\backslash(S_0 \cup D_0)$.

Step 2.

Choose a generalized vNM-stable set S_1 for (G,Y).
If $Y = \varnothing$, then $S_1 = \varnothing$.

Take $S = S_0 \cup S_1$.

Proof We first verify that the subset S constructed in the lemma above is a generalized vNM-stable set.

External stability: Suppose that $x \notin S$. If $x \in Max(\tau(G,X))$, then there is an alternative $y \in S_0$, such that yGx. If $x \in D_0$, the same is true. Now, suppose that $x \in Y$. Then, because S_1 is a generalized vNM-stable set of (G,Y), there is an element $y \in S_1$, such that yGx.

Generalized internal stability: Since $G|[S_0 \times S_1 \cup S_1 \times S_0] = \emptyset$, this is evident.

Now suppose that we have a generalized stable set S. We reconstruct S. Consider $S_0 = S \cap Max(\tau(G,X))$. Assuming that S_0 is not internal or external stable in $(G,Max(\tau(G,X)))$ easily leads to a contradiction. Hence S_0 is a generalized vNM-stable set for $(G,Max(\tau(G,X)))$. Now, elements of $X \backslash Max(\tau(G,X))$ that are directly dominated by S_0, included in the set D_0, cannot be element of S, since $S_0 \subseteq Max(\tau(G,X))$ and internal stability of S. Hence $S \backslash S_0 \subseteq Y = X \backslash (S_0 \cup D_0)$. Now we assert that $S \backslash S_0$ is a generalized vNM-stable set for (G,Y).

Generalized internal stability: Suppose $s,t \in S \backslash S_0$ and $s[\tau(G,S \backslash S_0)]t$. Then $s[\tau(G,S)]t$ hence $t[\tau(G,S)]s$. Let us write this as $t = z_0 G z_1 G \ldots G z_p = s$. Because $t = z_0 \notin Max(\tau(G,X))$ we know that $z_1 \notin Max(\tau(G,X))$. Continuing this way, $z_p = s \notin Max(\tau(G,X))$. Hence for all $i \in \{0,\ldots,p\}$ we have that $z_i \in S \backslash S_0$. Thus $t[\tau(G,S \backslash S_0)]s$.

External stability: Take $y \in Y \backslash S$. Then there does not exist an $s \in S_0$ such that sGy. But because S is a generalized stable set for (G,X), there is an $t \in S$ such that tGy, implying that $t \in S \backslash S_0$. ■

We illustrate the construction theorem using the first relation (G_5,X) of figure 4.4. In that case, $Max(\tau(G_5,X)) = \{a,b,c,d\}$, and there are three possible generalized vNM-stable sets for $(G_5,Max(\tau(G,X)))$: $\{a,b,c,d\}$, $\{a,b,c\}$ and $\{a,d\}$. The sets Y are respectively: $\emptyset$, $\{e\}$ and $\emptyset$. This gives three generalized vNM-stable sets $\{a,b,c,d\}$, $\{a,b,c,e\}$ and $\{a,d\}$, so we see that a generalized vNM-stable set is not unique.

In the remaining part of this subsection, we consider a few relations between the two modifications of vNM-stable sets.

First of all, we relate $Max(\tau(G,X))$ and $Max(G,X)$ to the modifications of the vNM-stable set.

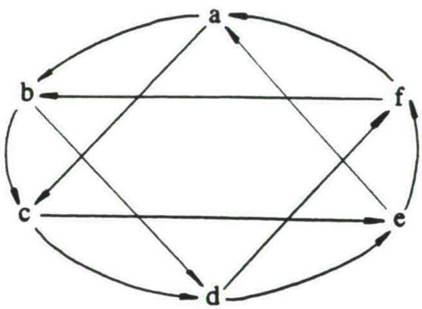
PROPOSITION 4.10 *Let $(G,X) \in \mathbb{G}(X)$. Then*

(a) $\cup\{D$: *D is a generalized stable set of* $(G,X)\} = Max(\tau(G,X)) \subseteq$
$\cup\{D$: *D is a generalized vNM-stable set of* $(G,X)\}$.

(b) $\cap\{D$: *D is a generalized stable set of* $(G,X)\} = Max(G,X) \subseteq$
$\cap\{D$: *D is a generalized vNM-stable set of* $(G,X)\}$.

Proof (a) The equality follows by theorem 4.17 in van Deemen (1991). The inclusion may be proved using the construction of a generalized vNM-stable set given in the construction theorem 4.9. (b) For the equality we refer to van Deemen (1991). The inclusion follows from the external stability. ■

Note that both intersections mentioned in the proposition above, may be empty. Take e.g. (G_7,X) to be equal to



Then $D_1 = \{a,c,e\}$ and $D_2 = \{b,d,f\}$ are two disjoint generalized vNM-stable sets of (G_7,X).

PROPOSITION 4.11 *Suppose (G,X) is complete, i.e. (G,X) is a tournament.*

(a) *If D_1, D_2 are generalized vNM-stable sets of (G,X), then $D_1 \cup D_2$ is a generalized vNM-stable set of (G,X).*

(b) $\cup\{D$: *D is a generalized vNM-stable set of* $(G,X)\}$ *is a generalized vNM-stable set of* (G,X).

(c) $\cup\{D$: *D is a generalized vNM-stable set of* $(G,X)\} =$
$\cup\{D$: *D is a generalized stable set of* $(G,X)\} = Max(\tau(G,X))$.

Proof (a) The external stability is no problem. Suppose aGb, where $a \in D_1$,

$b \in D_2$. Because D_2 is a generalized vNM-stable set, there is an element $d \in D_2$ such that dGa. If (G,X) is complete, b and d lie on a cycle in D_2, which means that there is a path from b to a in $D_1 \cup D_2$. (b). This is a corollary of (a). (c). The relation (G,X) is complete. Hence Max(τ(G,X)) is a generalized vNM-stable set. Moreover, because of generalized internal stability, all generalized vNM-stable sets are included in Max(τ(G,X)). ■

PROPOSITION 4.12 *If (G,X) is acyclic, then it has a unique generalized stable set and a unique generalized vNM-stable set. Both are equal to the unique vNM-stable set.* ■

4.4 BALANCED DOMINANT WEIGHTS

In this section, we refine the notions of generalized internal stability and external stability by the balanced dominant weights condition. This leads to a characterization of the choice function strat. Moreover, we show that there is a unique generalized vNM-stable set satisfying this condition.

As a generalization of theorem 4.2, we state theorem 4.13.

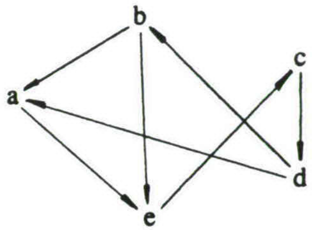
THEOREM 4.13 *For all asymmetric relations G: strat(G) is a generalized vNM-stable set of G.*

Proof Let Y = *strat*(G,X). We prove that Y is a generalized vNM-stable set.
External stability: If $x \notin Y$, then for all optimal strategies $\underline{p}$, we have $\underline{p}_x = 0$. Hence, for any $x \notin Y$, there exists an optimal strategy $\underline{q}$, such that $(\underline{q}A)_x > 0$. But this means that there exists an alternative $y \in \mathrm{supp}(\underline{q}) \subseteq Y$ such that yGx.
Internal stability: Let $x,y \in Y$ and xGy. Suppose that $\neg y[\tau(G,Y)]x$. We derive a contradiction.
Take an optimal strategy $\underline{p}$, such that $\mathrm{supp}(\underline{p}) = Y$. Then $\underline{p}$ is an optimal strategy for (G,Y). Let A be the restriction of the zero-sum matrix to Y^2. Consider the following partition of Y:
$S = \{z \in Y: z[\tau(G,Y)]x\} \cup \{x\}$ and $T = \{z \in Y: \neg z[\tau(G,Y)]x\}$ containing the alternative y. Note that for all $(t,s) \in T \times S$: $\neg tGs$. Hence, we may write

$$A = \begin{pmatrix} A|S^2 & D \\ C & A|T^2 \end{pmatrix},$$

where $C = (c_{ij})$, $c_{ij} \in \{-1,0\}$, $D = (d_{uv})$, $d_{uv} \in \{0,1\}$.
Because $\underline{p}$ is an optimal strategy, $\underline{p}A \geq \underline{0}$. Define $\underline{q} = \underline{p}|S$ and $\underline{r} = (\underline{q},\underline{0}) \in K^t$, $t = |Y|$. Then we have $\underline{r}A \geq \underline{0}$, because $\underline{p}A \geq \underline{0}$ and for all $(t,s) \in T{\times}S$: $\neg tGs$. Thus $\underline{r}$ is an optimal strategy for (G,Y). Moreover, $(\underline{r}A)_y > 0$ because xGy and $\underline{r}_x > 0$, which means that for all optimal strategies $\underline{v}$ of (G,Y), $\underline{v}_y = 0$, which is a contradiction, because $y \in \mathrm{supp}(\underline{p})$. ∎

As already mentioned before, generalized vNM-stable sets are not unique. Take again the relation of figure 4.2:



There are four generalized vNM-stable set: {a,b,c,d,e}, {b,c}, {d,e} and {b,c,d,e}. We show that just one of these sets is able to reach a satisfactory division of power.
If we take a look at the coalition {b,c}, we see that although they dominate {a,d,e}, it does not reflect the political practice of the formation of coalitions focused on the division of power between its members (and of course domination of the other parties involved). For, suppose that b and c do search for a division of power between them, (p_b,p_c) with $p_b + p_c = 1$. In order to dominate d, it is necessary that $p_c > p_b$. On the other hand, domination of e requires $p_b > p_c$. Thus b and c cannot reach a satisfactory division of power. The same is true for {d,e}. So, they cannot reach a division that is independent of the alternative that is to be dominated.
On the other hand, the coalition V = {b,c,d,e} is able to reach such a division of power, for example (1/4,1/4,1/4,1/4). This division has another

nice feature. For each $x \in V$, we may divide $V\setminus\{x\}$ into U_x and L_x. The set U_x is formed by the alternatives that dominate x, L_x is the set of alternatives that are dominated by x. For example, $U_b = \{d\}$, $L_b = \{e\}$. The division of power (1/4,1/4,1/4,1/4) has the property that for all x the sum of power of elements in U_x is equal to the sum of power of elements in L_x. It keeps the scale of power inside the coalition. Furthermore, everyone can tip the scale in his favour by using his own weight or power. Hence it is a dynamic coalition. Corollary 4.16 claims that of all generalized vNM-stable sets, just one, in our case $V = \{b,c,d,e\}$, is able to reach a division of power that is both internally and externally stable, in a refined meaning, or putting it otherwise, satisfies the balanced dominant weight condition, which we shall introduce next.

- **Balanced dominant weights**

  A choice function $S : \mathbb{G}(X) \to 2^X\setminus\{\varnothing\}$ satisfies the balanced dominant weights property, if for all $G \in \mathbb{G}(X)$ the following holds:
  $S(G)$ is support of a distribution of weights $\underline{p} \geq \underline{0}$ summing to 1, such that:

  $$(I)\quad \text{if } x \in S(G) \text{ then} \qquad \sum_{y\in S(G),\, yTx} \underline{p}_y = \sum_{y\in S(G),\, xTy} \underline{p}_y$$

  $$(II)\quad \text{if } x \notin S(G) \text{ then} \qquad \sum_{y\in S(G),\, yTx} \underline{p}_y > \sum_{y\in S(G),\, xTy} \underline{p}_y.$$

We first state a characterization of *strat*. Compare theorem 2.10 of chapter 2.

THEOREM 4.14 (Characterization of *strat*) *Let S be a choice function on $\mathbb{G}$. Then S = strat, iff S satisfies the condition of balanced dominant weights.*

Proof (if) Suppose $\underline{p}$ satisfies (I) and (II). Let $Y = S(G,X)$ be the support of $\underline{p}$. We prove that $Y = strat(G,X)$.
Because of (I), (II) and $Y = \mathrm{supp}(\underline{p})$, $\underline{p}$ is an optimal strategy: $(\underline{p}A) \geq \underline{0}$. Thus $Y \subseteq strat(G,X)$.
Conversely, if $x \in strat(G,X)\setminus Y \neq \varnothing$, then (II) implies that $(\underline{p}A)_x > 0$. But then, $\underline{q}_x = 0$ for all optimal strategies $\underline{q}$, which is a contradiction, because $x \in strat(G,X)$.

(only if) In general, we will construct an optimal strategy satisfying (II) and support equal to *strat*(G,X).

For all $x \in Y :=$ *strat*(G,X), there is an optimal strategy $\underline{r}$ such that $\underline{r}_x > 0$. Hence, there exists an optimal strategy $\underline{q}$ such that $\mathrm{supp}(\underline{q}) = Y$.

If $x \notin Y$, then for all optimal strategies $\underline{t}$, we know that $\underline{t}_x = 0$. But then, using a general theorem from game theory, for all $x \notin Y$, there is an optimal strategy $\underline{s}_x$, such that $(\underline{s}_x A)_x > 0$.

Define

$$\underline{w} = \sum_{x \in X \setminus Y} \lambda_x \underline{s}_x,\ \lambda_x \in (0,1) \text{ for } x \notin Y \text{ and } \sum_{x \in X \setminus Y} \lambda_x = 1.$$

Then $\underline{w}$ is an optimal strategy. Moreover, if $x \notin Y$ then $(\underline{w}A)_x > 0$, because $(\underline{s}_x A)_x > 0$ and $(\underline{s}_y A)_x \geq 0$, $y \neq x$.

Now define $\underline{p} = \lambda\underline{q} + (1-\lambda)\underline{w}$, $\lambda \in (0,1)$. Then $\underline{p}$ is an optimal strategy, $\mathrm{supp}(\underline{p}) = Y$, and if $x \notin Y$ then $(\underline{p}A)_x > 0$. Hence $\underline{p}$ also satisfies (II). ■

Analogous to proposition 4.7, we have

PROPOSITION 4.15 *Take $(G,X) \in \mathbb{G}(X)$. Then $\underline{p}$ satisfies the conditions of theorem 4.14, iff $[O(G) = \{\underline{p}\}$ or $\underline{p} \in int(rel(O(G)))]$.* ■

So, this set of distributions of weights is convex, and it is open in rel(O(G)) if $|O(G)| > 1$.

Note that if S is a vNM-stable set, then S is support of an optimal strategy of the G-game. Therefore, the necessity of the condition in the following conjecture is easily proved.

CONJECTURE *Let $(G,X) \in \mathbb{G}(X)$. Then there is a unique optimal strategy for (G,X), iff*

(i) $Max(\tau(G,X))$ is a generalized vNM-stable set

(ii) If V is a vNM-stable set of (G,X), then V is a singleton. ■

Using theorems 4.13 and 4.14, we may state that *strat*(G,X) is the only generalized vNM-stable set that satisfies the balanced dominant weight condition. In corollary 4.16, (I) prescribes possible divisions of power leading to a standing coalitions, while (II) expresses the corresponding decisiveness.

COROLLARY 4.16 *Let $(G,X) \in \mathcal{G}(X)$. Then strat(G,X) is the unique generalized vNM-stable set V that is the support of a distribution of weights $\underline{p} \geq \underline{0}$ summing to 1, satisfying*

$$(I) \quad \text{if } x \in V \text{ then} \quad \sum_{y \in V,\ yGx} \underline{p}_y = \sum_{y \in V,\ xGy} \underline{p}_y$$

$$(II) \quad \text{if } x \notin V \text{ then} \quad \sum_{y \in V,\ yGx} \underline{p}_y > \sum_{y \in V,\ xGy} \underline{p}_y. \quad \blacksquare$$

The notion *strat* reflects the political practice of the formation of coalitions focused on the division of power between its members, and domination of the other parties involved, which is formally expressed in the corollary given above.

In the example of figure 4.2 discussed above, all optimal strategies are equal to $\underline{p} = \lambda(0,1/2,1/2,0,0,0) + (1-\lambda)(0,0,0,1/2,1/2)$. But, as one may easily verify, $\underline{p}$ satisfies the conditions of corollary 4.16, if and only if $\lambda \in (0,1)$, meaning that $\{b,c,d,e\}$ is the unique generalized vNM-stable set mentioned in that corollary. In general, the distribution $\underline{p}$ over Y is not unique. The same example shows that this unique generalized vNM-stable set is not necessarily the largest generalized vNM-stable set, which equals $\{a,b,c,d,e\}$.

4.5 CHOICE FUNCTIONS FOR DOMINANCE RELATIONS

In subsection 4.5.1, we extend a few choice functions introduced in section 4.2 to choice functions on arbitrary asymmetric relations. We assume that the outcomes in (G,X) represent domination. We verify some properties for our choice functions that are significant in this context. In section 4.5.2, we study ranking rules that are induced by the choice functions of section 4.5.1.

4.5.1 Choice functions

We consider four choice functions on $\mathbb{G}(X)$.

vNM, vNM-max, top, strat

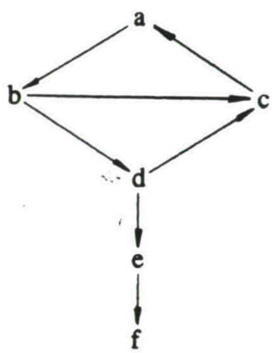
Let $(G,X) \in \mathbb{G}(X)$. Then $vNM(G,X)$ is defined to be the union of all generalized vNM-stable sets.

For $(G,X) \in \mathbb{G}(X)$, we let $vNM\text{-}max(G,X)$ be that generalized vNM-stable set that is obtained by taking $S = Max(\tau G,Y)$ at each recursively defined step in theorem 4.9.

For all (G,X), we let $top(G,X)$ be equal to $Max(\tau(G,X))$.

For the definition of *strat*, we refer to chapter 2 section 6 or section 4.2 of this chapter.

To give an example, consider the following relation (G_8,X), where $X = \{a,b,c,d,e,f\}$.



Now, $top(G_8,X) = \{a,b,c,d\}$, $vNM(G_8,X) = X$, $vNM\text{-}max(G_8,X) = \{a,b,c,d,f\}$ and $strat(G_8,X) = \{a,b,c,d,f\}$. Possible distributions of weight over $strat(G_8,X)$ in correspondence with theorem 4.14 of section 4.4 are equal to $\lambda_1(1/2,0,1/2,0,0,0) + \lambda_2(1/3,1/3,0,1/3,0,0) + \lambda_3(1/3,0,1/3,0,0,1/3)$, where $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $0 < \lambda_i < 1$ for $i \in \{1,2,3\}$.

Independence conditions

We consider some properties in relation to the choice functions presented above. These properties were introduced and discussed in chapter 2.

THEOREM 4.17 *The choice function strat on asymmetric relations*

(a) satisfies Δ-IIA,

(b) is tail stable,

(c) satisfies SSP,

(d) is globally monotone.

Proof Let Y be the unique set that is support of a distribution of weights satisfying conditions (I) and (II) of theorem 4.14. Since in all four cases (a), (b), (c) and (d), nothing changes at the expense of this set, it is easy to verify that the distribution of weights, as described in theorem 4.14, still satisfies these conditions. Using the characterization of *strat*, the proof follows.∎

The same properties as in proposition 4.17 hold for *vNM-max* and for *vNM*. The proof for *vNM-max* is simple and is left to the reader. For the proof of *vNM*, we need the following lemma, see figure 4.5.

LEMMA 4.18 *Take $(G,X) \in \mathbb{G}(X)$. Let S be any generalized vNM-stable set of (G,X). Then, for all $y \in X$, there is an $m \in Max(\tau(G,X))$, such that there exists a path via G from m to y, such that it never occurs that two consecutive alternatives on that path are both element of $X \setminus S$.*

Proof The proof is easy, if we repeatedly use the construction theorem 4.9: for S ($S = S_0 \cup S_1$), for S_1 ($S_1 = S_{10} \cup S_{11}$), for S_{11}, etc. ∎

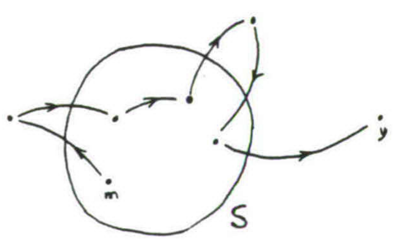


Figure 4.5 Illustration of lemma 4.18.

THEOREM 4.19 *The choice function vNM*

(a) satisfies Δ-IIA,

(b) is tail stable,

(c) satisfies SSP,

(d) is globally monotone.

Proof (a) Let $(G,X) \in \mathbb{G}(X)$ and $x \in X$. Suppose $vNM(G,X_x) = D$. Hence D is the union of all generalized vNM-stable sets of (G,X_x). To prove that *vNM* is Δ-IIA, we assume that for all $d \in D$, dGx. Now, it easy to verify that a generalized vNM-stable set of (G,X_x) is so for (G,X). Moreover, it is clear that $\mathrm{Max}(\tau(G,X_x)) = \mathrm{Max}(\tau(G,X))$, so $\mathrm{Max}(\tau(G,X)) \subseteq D$.

Suppose S is a generalized vNM-stable set of (G,X). Because in general all generalized vNM-stable sets do contain at least one alternative from $\max(\tau(G,X))$, we may take $x_T \in \mathrm{Max}(\tau(G,X)) \cap S \neq \emptyset$. Suppose that $x \in S$. Then, since $x_T G x$, the internal stability implies that $x(\tau G)x_T$, implying that $x \in \mathrm{Max}(\tau(G,X))$. Since $\mathrm{Max}(\tau(G,X)) \subseteq D$, this gives a contradiction. So, $x \notin S$. But then we deduce that S is a generalized vNM-stable set of (G,X_x).

(b) We use induction to $k = |X|$.

Induction hypothesis. For all X such that $|X| \leq k$, the following holds. For all $(G,X) \in \mathbb{G}$, if $vNM(G,X) = N$ and (G',X) is such that differences between G' and G are limited to $(X\backslash N)^2$, then $vNM(G,X) = vNM(G',X)$.

The case $k = 1$ is obvious.

Take $(G,X) \in \mathbb{G}(X)$, with $|X| = k + 1$. Let $N = vNM(G,X)$, and let G' be as described above.

First we prove $\mathrm{Max}(\tau(G',X)) = \mathrm{Max}(\tau(G,X))$.

($\supseteq$). Suppose $z \in \mathrm{Max}(\tau(G,X)) \subseteq vNM(G,X)$ and $t\tau G'z$. This means there do exist $t_0, \ldots, t_k$ such that $t = t_0 G' t_1 G' t_2 G' \ldots t_{k-1} G' t_k = z$. But $t_{k-1} G' z$ means $t_{k-1} G z$, and $t_{k-1} \in \mathrm{Max}(\tau(G,X))$. Analogously: $t_i \in \mathrm{Max}(\tau(G,X))$ for all i. Thus $t \in \mathrm{Max}(\tau(G,X))$. But then $t\tau Gz$, which implies $z\tau Gt$, where we may restrict G to $\mathrm{Max}(\tau(G,X))$. Therefore we have $z\tau G't$, and thus we obtain that $z \in \mathrm{Max}(\tau(G',X))$.

($\subseteq$). Take $z \in \mathrm{Max}(\tau(G',X))$. There exists an $x_T \in \mathrm{Max}(\tau(G,X))$ such that

$$x_T \tau G' z \qquad \text{(lemma 4.18)}$$

hence

$$z \tau G' x_T \qquad (z \in \mathrm{Max}(\tau(G',X)))$$

so

$$z\tau Gx_T \qquad \text{(by choice of G' and } x_T \in \mathrm{Max}(\tau(G,X)))$$

which means

$$z \in \mathrm{Max}(\tau(G,X)) \qquad (x_T \in \mathrm{Max}(\tau(G,X)).$$

Hence, indeed, $\mathrm{Max}(\tau(G',X)) = \mathrm{Max}(\tau(G,X))$.

From the construction theorem 4.9, we may deduce that

$$vNM(G,X) = \bigcup_S \{S \cup vNM(G,X\setminus[S \cup D_S(G)])\},$$

where S is a generalized vNM-stable set of $(G,\mathrm{Max}(\tau(G,X)))$ and $D_S(G) = \{y \in X\text{: there is an } s \in S \text{ such that } sGy\}$.

Using that $\mathrm{Max}(\tau(G,X)) = \mathrm{Max}(\tau(G',X))$ and the fact that differences between G and G' are limited to $(X\setminus N)^2$, we may state:
S is a generalized vNM-stable set of $(G,\mathrm{Max}(\tau(G,X)))$ iff S is so for $(G',\mathrm{Max}(\tau(G',X))) = (G,\mathrm{Max}(\tau(G,X)))$ and, moreover, $D_S(G) = D_S(G')$.
Since $vNM(G,X\setminus[S \cup D_S(G)]) \subseteq N$, we may use the induction hypothesis, to deduce that $vNM(G,X\setminus(S \cup D_S(G))) = vNM(G',X\setminus(S \cup D_S(G')))$, which completes the proof.

(c) Let $x \notin vNM(G,X)$. Consider $vNM(G,X_x)$, where $X_x = X\setminus\{x\}$. It is clear that $\mathrm{Max}(\tau(G,X)) \subseteq \mathrm{Max}(\tau(G,X_x))$. Next, let us assume that $\mathrm{Max}(\tau(G,X_x))\setminus\mathrm{Max}(\tau(G,X)) = K$, where K consists of elements on cycles C in (G,X), which are unbeaten in (G,X_x).
Since $G|[\mathrm{Max}(\tau(G,X)) \times K] \cup [K \times \mathrm{Max}(\tau(G,X))] = \varnothing$, we may state

$vNM(G,X_x) = \bigcup_S \{S \cup vNM(G,X_x\setminus[S \cup D_S(G)])\}$, where S is a generalized vNM-stable set of $(G,\mathrm{Max}(\tau(G,X)))$. Because $x \notin vNM(G,X\setminus[S \cup D_S(G)])$, we may use induction, analogous to the proof of (b), to complete the proof of the assertion that *vNM* satisfies SSP.

(d) Analogous to (b) if we use the result from (c). ∎

THEOREM 4.20 *The choice function top*

(a) is Δ-IIA

(b) is globally monotone. ∎

As shown in section 4.2, it is not tail stable. It also does not satisfy SSP.

4.5.2 Induced ranking rules

We introduce four ranking rules, induced by the choice functions *vNM*, *top*, *strat* and *vNM-max*: f_{vNM}, f_{top}, f_{strat} and $f_{vNM\text{-}max}$.
We repeat the definition of an induced ranking rule.

Let S be a choice function on $\mathbb{G}$. *then the induced ranking rule* f_S *on* $\mathbb{G}$ *is defined as follows. For all* $(G,X) \in \mathbb{G}(X)$,

$$\text{if } X\setminus S(G,X) = \emptyset \text{ then } f_S(G,X) = S(G,X),$$
$$\text{else } f_S(G,X) = S(G,X) \gg f_S(G,X\setminus S(G,X)).$$

First of all, we have

PROPOSITION 4.21 *Restricted to tournaments,* $f_{vNM} = f_{top} = f_{vNM\text{-}max} = f_{trans}$, *which is the transitive closure.*

Proof Let $f_{trans}(T,X) = X_1 \gg \ldots \gg X_n$ where the X_i are the irreducible parts of (T,X) and $(T,X) = (T,X_1) \gg \ldots \gg (T,X_n)$. We prove that $vNM(T,X) = vNM\text{-}max(T,X) = top(T,X) = X_1$.
Now, $X_1 = \mathrm{Max}(\tau(T,X))$. Moreover, X_1 is a generalized vNM-stable set and, because of internal stability, all generalized vNM-stable sets are included in X_1. Hence $vNM(T,X) = vNM\text{-}max(T,X) = top(T,X) = X_1$.
Because for $i \in \{2,\ldots,n\}$, $X_i = \mathrm{Max}[\tau(T,X\setminus(X_1 \cup \ldots X_{i-1}))]$, the remaining parts of the proof are analogous. ■

Next, we verify some independence conditions, that are analogous to the properties in the previous subsection.

First of all, we consider the tail stability for ranking rules. For this purpose, we need the substitution operation and the notion of a right-interval, both introduced in chapter 3.

Let $Y \subseteq X$, G,G' arbitrary binary relations on X.
Then $\mathrm{Sub}((G,X),(G',Y)) := \{(x,y) \in X\times X :$
$((x,y) \in Y\times Y \cap G')$ or $((x,y) \notin Y\times Y$ and $(x,y) \in G)\}$.

Let $W = X_1 \gg \ldots \gg X_n$ be a weak order. A right-interval of W is a set $Y = X_i \cup \ldots \cup X_n$, $i \in \{1,\ldots,n\}$.

A ranking rule f on $\mathbf{G}$ *is tail stable, if for all* $(G,X) \in \mathbf{G}$ *and all right-intervals Y of* $f(G,X)$ *and all* $(G',Y) \in \mathbf{G}$:

$$f(Sub((G,X),(G',Y))) = Sub(f(G,X),f(G',Y)).$$

The second condition that we consider is the global monotonicity.

A ranking rule f on $\mathbf{G}$ *is globally monotone, if the following is true. Let G be an asymmetric relation on X. Suppose* $f(G) = X_1 \gg \ldots \gg X_n$. *Let* $x,y \in X$ *be such that* $x \in X_i$, $y \in X_j$, $j > i$. *Then* $f(G) = f(G')$, *if G' is such that* $G' = [G\backslash\{(y,x)\}] \cup \{(x,y)\}$,

This condition can be interpreted in terms of domination. Suppose that y, who first dominates x, agrees with the fact that x dominates y in the ranking given by *f*. This means that yGx is overruled by global insight. If we express this in the relation G', we do not want the ordering to change.

PROPOSITION 4.22 *(a) The rules* f_{vNM}, $f_{vNM\text{-}max}$, f_{top} *and* f_{strat} *on* $\mathbf{G}$ *are* Δ*-IIA and globally monotone. (b) The rules* f_{vNM}, $f_{vNM\text{-}max}$ *and* f_{strat} *on* $\mathbf{G}$ *are tail stable.*■

Proposition 4.23 to 4.25 study the relation between choice functions and induced ranking rules.

PROPOSITION 4.23 *The set of ranking rules f on* $\mathbf{G}$ *that are tail stable, is a subset of the set of ranking rules induced by a choice function.*

Proof Define $S(G,X) = Best(f(G,X))$ for all $(G,X) \in \mathbf{G}$. Now, let $f(G,X) = X_1 \gg \ldots \gg X_n$, so $S(G,X) = X_1$. Take $Y = X_2 \cup \ldots \cup X_n$. Of course, we have $(G,X) = Sub((G,X),(G,Y))$. Because *f* is tail stable, $f(G,X) = Sub(f(G,X),f(G,Y)) = X_1 \gg f(G,Y) = S(G,X) \gg f(G,X\backslash S(G,X))$. ■

In chapter 3, we proved: if *f* is interval consistent, then it is induced by a choice function. Conversely, we have

PROPOSITION 4.24 *If a choice function S on* G *satisfies SSP, then f_S is interval consistent.*

Proof Take $G \in \mathbf{G}(X)$ and let $f_S(G) = X_1 \gg \ldots \gg X_n$. Take an interval $I = X_i \cup \ldots \cup X_j$. Now, $f_S(G,I) = S(G,I) \gg f_S(G,(I \setminus S(G,I)))$, because f_S is induced by S. Furthermore, $X_i = S(G,X_i \cup \ldots \cup X_n)$, and because S satisfies SSP, we have $S(G,X_i \cup \ldots \cup X_n) = S(G,I)$. Hence,

$$f_S(G,I) = X_i \gg f_S(G,X_{i+1} \cup \ldots \cup X_j).$$

The remaining parts of the proof that $f_S(G,I) = f_S(G)|I$ are analogous. ■

We also can prove the following result.

PROPOSITION 4.25 *If a ranking rule f on* G *is globally monotone and commutes with concatenation, then f is induced by a choice function S on* G.
If moreover, f is tail stable, then S satisfies SSP.

Proof Let $G \in \mathbf{G}(X)$ and suppose that $f(G) = X_1 \gg \ldots \gg X_n$. Take an interval $I = X_i \cup \ldots \cup X_j$. We prove that $f(G,I) = f(G)|I$. Consider $(G',X) =$

$$(G,X_1) \gg \ldots \gg (G,X_{i-1}) \gg (G,I) \gg (G,X_{j+1}) \gg \ldots \gg (G,X_n).$$

The ranking rule f being globally monotone, we may deduce that $f(G',X) = f(G,X)$. But, since f commutes with concatenation, $f(G',X) =$

$$f(G,X_1) \gg \ldots \gg f(G,X_{i-1}) \gg f(G,I) \gg f(G,X_{j+1}) \gg \ldots \gg f(G,X_n),$$

implying that $f(G,I) = X_i \gg \ldots \gg X_j = f(G)|I$. Hence, f is interval consistent. In chapter 3, we proved that if f is interval consistent, then it is induced by a choice function S. Next, we assume that f is tail stable. Suppose that $S(G,B') \subseteq B \subseteq B'$. We have to prove that $S(G,B) = S(G,B')$. Because f is globally monotone and is tail stable, we may change the relation (G,B') to $(G',B') = (G,B) \gg (G,B' \setminus B)$, while $S(G',B') = S(G,B')$. Since f commutes with concatenation, S is concatenation consistent, hence $S(G',B') = S(G,B)$. ■

We next state a characterization of f_{top}.

THEOREM 4.26 (Characterization of f_{top}) *Let f on* G *be a ranking rule. Then* $f = f_{top}$ *iff*

- *f is globally monotone*
- *f commutes with concatenation*
- $Max(f(G,X)) = Max(\tau(G,X))$ *for all* $(G,X) \in$ G.

Proof (only if) The global monotonicity of f_{top} follows from theorem 4.22. The other two properties are evident. (if) In proposition 4.25, it is proved that the first two properties imply that f is induced by a choice function. The third property shows which choice function: *top*. ■

COROLLARY 4.27 *Let f be a ranking rule. Then* $f = f_{top}$ *iff*

- *f is interval consistent*
- $Max(f(G,X)) = Max(\tau(G,X))$ *for all* $(G,X) \in$ G. ■

We end this section with a characterization of f_{vNM} restricted to acyclic asymmetric relations. It resembles theorem 4.5. To prove the theorem, we need the following lemma.

- **Externally consistent**

  Let f be a ranking rule on G. *Then f is said to be externally consistent, if the following holds for each* $(G,X) \in$ G.

  Suppose $f(G,X) = X_1 \gg \dots \gg X_n$. *Then for all* $1 \le i < j \le n$ *and all* $y \in X_j$, *there is an* $x \in X_i$, *such that xGy.*

LEMMA 4.28 *Let f be a ranking rule on* A, *the set of acyclic asymmetric relations. Suppose*

- *f is tail stable*
- $Max(G,X) \subseteq Max(f(G,X))$, *for all* $(G,X) \in$ A.

Then f is externally consistent.

Proof Assume that $f(G,X) = Z_1 \gg \dots \gg Z_k$. Take $1 \le i < k$. Suppose there is an $z \in Z_j$, $j > i$, such that for no $z_i \in Z_i$ we have $z_i Gz$ in (G,X). Define $Y = Z_i \cup \dots \cup Z_k$ and consider (G,Y). Of course, (G,X) = Sub((G,X),(G,Y)). Because f is tail stable, f(G,X) = Sub(f(G,X),f(G,Y)). Hence $f(G,Y) = f(G,X)|Y = Z_i \gg \dots \gg Z_k$.
Now define $V = Z_{i+1} \cup \dots \cup Z_k$. If we delete all relations vGz where $v \in V$,

obtaining $(G',V) \in \mathbb{A}(X)$, we see that z is top element of $Sub((G,Y),(G',V))$ which is acyclic. But it is clear that z is not a top element of $f(Sub((G,Y),(G,V))) = Sub(f(G,Y),f(G,V))$, which contradicts the second assumption. ■

THEOREM 4.29 (Characterization of f_{vNM} on $\mathbb{A}$) *Let f be a ranking rule on* $\mathbb{A}$. *Then* $f = f_{vNM}$ *iff*

- *f is tail stable*
- $Max(G,X) \subseteq Max(f(G,X))$
  (Condorcet principle)
- *if aGb then* $\neg a \sim b : f(G,X)$.
  (local discriminating ability)

Proof (only if) The first property is proposition 4.22(b). The second may be proved using the construction theorem 4.9. To prove the third one, we observe that because of acyclicity of (G,X) and internal stability of the unique vNM-stable set, a and b can not both be element of it. Suppose $f(G,X) = Z_1 \gg \ldots \gg Z_n$. Analogous arguments lead to the conclusion that for all $i \in \{1,\ldots,n\}$, if $a \in Z_i$ then $b \notin Z_i$.
(if) Suppose $f(G,X) = Z_1 \gg \ldots \gg Z_n$. From lemma 4.28 we deduce that the first two properties imply that f is externally consistent. For example, we know that for all $x \in X \setminus Z_1$ there does exist an element $z \in Z_1$ such that zGx. Combining this with the third property, we deduce that Z_1 is the unique vNM-stable set. Because f is tail stable on acyclic relations, we may deduce that Z_2 is the unique generalized vNM-stable set of $X \setminus Z_1$. Indeed, $f(G,X) = Z_1 \gg Z_2 \gg \ldots \gg Z_n$. Define $Y = X \setminus Z_1$. Then $f(Sub((G,X),(G,Y))) = Sub(f(G,X),f(G,Y))$. Hence $f(G,Y) = Z_2 \gg \ldots \gg Z_n$. In an analogous way we deduce that for $i \in \{3,\ldots,n\}$, Z_i is the unique generalized vNM-stable set for $X \setminus (Z_1 \cup \ldots \cup Z_{i-1})$. ■

The logical independence of the conditions:

(1) Define f_t as follows. $a \gg b : f_t(G,X)$ iff there exist a partition $X = A \cup B$ such that $a \in A$, $b \in B$ and $(G,X) = (G,A) \gg (G,B)$. It is fairly easy to show, using theorem 3.4 of chapter 3 that $h = f_t$ iff h is independent of interval changes and commutes with concatenation and permutation. Hence f_t satisfies condition 1 and 2, but violates condition 3.

(2) Define $f^{\nu}_{vNM}(G,X) := f_{vNM}(\nu G,X)$. This rule satisfies conditions 1 and 3, but clearly violates condition 2.

(3) The rule f_{top} satisfies condition 2 and 3. Condition 1 is violated.

We conclude the chapter with two tables.

	vNM	*vNM-max*	*top*
vNM-max	$\subseteq$		
top	$\subseteq$	$\subseteq$	
strat	$\subseteq$	$\cap$	$\cap$

Explanation: (i) $S \subseteq S'$ iff for all $G \in \mathcal{G}$, $S(G) \subseteq S'(G)$

(ii) $S \cap S'$ iff for all $G \in \mathcal{G}$, $S(G) \cap S'(G) \neq \emptyset$.

Table 4.1 Set-theoretical relations.

	Δ-IIA	tail stability	global monotonicity	SSP	balanced dominant weights
vNM	■	■	■	■	□
vNM_{max}	■	■	■	■	□
top	■	□	■	□	□
strat	■	■	■	■	■

□ : not satisfied

■ : satisfied

Table 4.2 Table of properties

CHAPTER 5

CIRCULARITY MEASURES FOR TOURNAMENTS

5.1 INTRODUCTION

Decision theory is partly aimed at structuring and clarifying subjective reasoning. One aspect of this is the consistency or reliability of a set of pairwise comparisons. This may be seen as the degree to which a set of alternatives can be ordered from best to worst or to what extent a preference relation can be sorted out.

Therefore, in all qualitative decision models there does exist a possibility for the determination of the circularity or inconsistency. If the circularity is too high, the problem must be reconsidered. Indeed, this high circularity may indicate that the decision maker uses unsuitable criteria in the evaluation of the alternatives. There may also be other factors, such as dependence between criteria, limited reasonal skills or the simple fact that all alternatives are more or less equal attractive.

For tournaments, there is a variety of circularity measures that one may use. Well-known examples are Slater's i and the number of 3-cycles. But this variety also appears to be a problem, because in general, the measures give different answers. Furthermore, there does not exist a circularity measure which is recognized or generally accepted as being the best. Therefore, one tries to find measures which are most suitable for a certain class of problems. This idea is carried out in this chapter by means of characterizations and statistical analysis for some measures, thereby indicating their domain of applicability.
We start with some definitions.

Definition 5.1
A circularity measure γ on $\mathbb{T}(X)$ is a function $\gamma : \mathbb{T}(X) \to \mathbb{R}$. A circularity measure γ on $\mathbb{T}$ is a family of circularity measures, one for each $\emptyset \neq X \subseteq U$.

We mention a few publications concerning circularity measures.

Bezembinder (1981,1991), Coombs (1958), Davids (1988), Kendall-Smith (1940), Phillips (1969), Slater (1961), Maas, Bezembinder, Wakker (1991).

5.2 INTRANSITIVITY AND RATIONALITY

In this section we discuss intransitive decisions and rationality. We show that in some cases, it is possible for a person to reveal intransitive decisions, while being rational. Nevertheless, in most cases, circularity measures appear to be useful: they measure to what extent a set of preferences can be sorted out, thereby giving an indication of a person's consistency.

5.2.1 Definition and examples of intransitive preference

Rationality often is subdivided into two parts: (a) internal consistency (e.g. transitivity) and (b) pursuit of self-interest (Pareto optimallity etc). As noted by Sen (1986), this definition is too mechanical and too permissive and does not capture the content of rationality. For example, a person is considered to be rational when his preferences are transitive. But few people would consider someone to be rational if the choices are made by using the transitive, reversed preferences. Also, we cannot say that someone not pursuing his own interest is not rational. He may serve another goal. According to Sen, rationality is the correspondence between the actual choice and a persons reasoning and quality of that reasoning.

In Delver, Monsuur and Storcken (1991), we followed another route in the investigation of the phenomenons of rationality and intransitivity. There we started from the principle that alternatives as well as value systems may be clarified to such a degree, that between any two elements of X (the set of alternatives), a strict preference is uncovered. In addition, we assume that, after a certain threshold has been passed, adding further detail will not, in any of the comparisons, cause reversal of preference.

X is value-distinguishable to a given subject if in each direct comparison between two of its elements, their context and attributes on the one hand, and objectives of choice and the subject's sense of value on the other hand, have been sufficiently clarified, to reveal a definitive strict *preference of one of the alternatives over the other. (Delver et al. 1991).*

This notion allows for intransitive decisions or preferences:

An intransitivity is the result of (genuinely) intransitive preference, if there is no reason to believe that the binary decisions will change under reconsiderations, because the set of alternatives X is value-distinguishable and the intransitivities are compatible with a theory under which the choices are being studied (compare Bezembinder and van Acker, 1980).

The combination of value-distinguishability and cycles may seem irrational. But as mentioned in Delver et al. (1991), actually two levels of analysis are involved. *A tournament is value-distinguishable at the binary relation level or local level, while cycles occur in the tournament as a whole, the global level.* Hence, alternatives may be value-distinguishable and yet be equivalent.
Before we illustrate this idea, we may, in a non-specialist way of speaking, compare this with some drawings from the famous Escher, for example, 'waterfall'. *Locally*, at every place, the drawing gives a (correct) two-dimensional projection of a three-dimensional space. But *globally*, the drawing belongs to the well-known class of impossible figures.

Now, suppose you own a restaurant. At a certain moment there are 3 guests, all who want to have the same dinner. You can offer them the choice between dinners a, b and c. The three guests reveal the following preference:

guest 1: a » b » c
guest 2: b » c » a
guest 3: c » a » b.

Which dinner has to be served? Looking at the majority in each pairwise comparison of the available dinners, you will reach the decision: a » b, b » c and c » a, a cyclic pattern, see figure 5.1. To Fishburn (1970), it illustrates the untenability of the transitivity condition as a general desideratum for social choice functions.

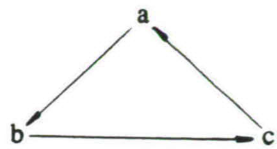


Figure 5.1 A circular tournament.

The second example takes a closer look at what you do not choose in a pairwise comparison, by introducing the notion of regret, which is the only decision criterion in the following example. Let X = {a,b,c} where the elements refer to actions in the decision table below, with states P, Q and R, each having probability 1/3. The numbers in the table are payoffs.

	P	Q	R
a	0	2	1
b	1	0	2
c	2	1	0

Table 5.1 Decision table, (compare Loomes and Sugden, 1984, page 231).

Using Savage's minimax regret criterion in the pairwise comparisons, the subject would find X value-distinguishable: The maximum regret of b with respect to a is 2, which would occur if Q materializes. Conversely, the maximum regret of a relative to b is 1, in states P and R. If one tries to minimize the maximal possible regret, a is preferred to b. Similarly b is preferred to c and c to a. All the same a, b and c form a cycle, see figure 5.1. The value of what is chosen depends on what has been rejected.

These examples show that it is not always possible to obtain transitivity after careful reflection. Intransitivity is not experienced as the result of poor reasoning. Indeed, in both examples, the alternatives are value-distinguishable and the decisions are compatible with a theory of choice.

The two examples clearly illustrate the need for a global level of analysis. Indeed, from a more global viewpoint, we see that the alternatives 'play the same role'. This may be seen with the help of figure 5.1: there does not exist a reasonable scale of measurement that can treat the three alternatives as being different. Hence, instead of circulating forever between the three possible choices, each time striving for a better alternative, we switch to a higher level of analysis. We conclude that the three alternatives are equivalent.

In the scientific literature, one may find other examples of intransitive choices. For example, if the alternatives are multi-dimensional, violations of transitivity may be observed. The number of primitive characteristics (attributes) by which alternatives can be compared is so large as to necessitate a selection of the criteria. Since this sampling may depend on the alternatives that are to be compared, there is a possibility of circular behaviour. See Quandt (1956). Also see some experiments conducted by Tversky (1969). He showed that under specified experimental conditions, consistent and predictable intransitivities can be demonstrated.

Yet another example of an intransitive phenomenon is given by the intransitive indifference. If R is a preference on a set X, then indifference I is defined to be

$$aIb \Leftrightarrow (\text{not } aRb) \text{ and } (\text{not } bRa).$$

Luce (1956) supplies arguments against the transitivity of indifference. Most people would prefer a cup of coffee with one spoon of sugar to a cup with five spoons. But if sugar is added by 1/100 of a gram, they would be indifferent between successive cups. This contradicts the transitivity of indifference. Luce therefore introduced the concept of a semi-order. Motivated by the use of thresholds in psychophysics, he gave the following representation of a semi order:

$$aRb \Leftrightarrow \quad \text{there exists an } f \text{ such that } f(a) > f(b) + \delta,$$

where δ is a pre-chosen constant > 0.

In Fishburn (1991), examples, showing why a cyclic pattern xTy, yTz and zTx may be rejected, are discussed. They all concern situations in which the intransitivity, together with other principles of rational decision making, leads to contradictions. For example, consider the following actions f and g in a three-state context with equally likely states 1, 2 and 3:

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline f & x & y & z \\ g & y & z & x \end{array}$$

The state-by-state dominance principle says that f is preferred to g, because xTy, yTz and zTx. But, on the other hand, f and g results in precisely the same lottery: you receive price x, y and z each with probability 1/3.
To Fishburn this problem is not real. He argues that the strong dominance is more compelling than reduction, because reduction separates the comparison from its context by obliterating the state-by-state alignments of outcomes for the two actions. Hence, $f \gg g$ is considered to be reasonable in this example.
In my opinion, again two levels of analysis may be distinguished, a local and a global one. The state-by-state comparison yields $f \gg g$, the global analysis leads to the conclusion that $f \cong g$. This is no contradiction, because the conclusions are not objectively comparable and, in our view, cannot be made independent of the kind of analysis in which one is involved or theory one adheres to. An analogous situation occurs in studying the phenomenon of light: the nature of light (wave or particle) depends upon the setting of the experiment.

Because of the occurrence of intransitivities, many authors reject the weak order as part of a normative theory, which deals with ideals and principles (often culturally conditioned) of good decision making. Fishburn (1991) mentions three lines of research that challenge the status of transitivity as cornerstone of order and rationality. Firstly, a variety of experiments and examples suggest that people sometimes violate transitivity, like in the examples above or as described in Tversky (1969). Secondly, theoretical studies show that in many cases one may obtain Nash equilibria, maximally preferred lotteries, economic equilibria and so on, without the assumption of transitivity. Thirdly, new models have been developed and axiomatized that do not assume transitivity. Most of these models are in the form

$$x \geq y \iff \Phi(x,y) \geq 0,$$

where $\Phi : XxX \to \mathbb{R}$. A cycle xTy, yTz and zTx is modeled by $\Phi(x,y) > 0$, $\Phi(y,z) > 0$ and $\Phi(z,x) > 0$. If $\Phi(x,y) = u(x) - u(y)$, we obtain the familiar representation

$$x \geq y \iff u(x) \geq u(y).$$

Also see van Acker (1977).

5.2.2 On the use of circularity measures

Although nontransitive models do exist, the concept of an order has attractive properties. It is easy to use and easy to understand. Moreover, practical considerations may force us to make choices in correspondence with the assumption of transitivity. For example, a politician looses his credibility if his priorities are circular.

But, as demonstrated in the previous subsection, tournaments often contain intransitivities. Therefore, these tournament results have to be ordered. This deciding between alternatives using tournament results, means the balancing of global information. Circularity measures can be indicative in that process. This way, using circularity measures, we think that we can help a decision maker to explore his problem and bring inconsistent beliefs and preferences to his attention so that he can resolve them.

5.3 SLATER'S I AND KENDALL-BABINGTON SMITH'S LAMBDA

We consider two well-known circularity measures: Slater's i and the number of 3-cycles λ, which was introduced by Kendall and Babington Smith. In section 5.3.1, we introduce them and show some differences. In section 5.3.2 it is shown that a combination of a few properties leads to an impossibility. Altogether, these two sections illustrate the need for characterizations. These are developed in sections 5.3.3 and 5.3.4. Finally, in section 5.3.5, we give a discussion. (In section 5.6, we use statistical techniques to compare λ and i.)

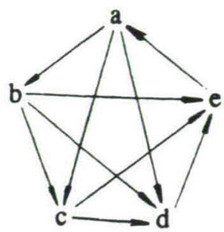
5.3.1 Introduction of the two circularity measures

In this section, we introduce two circularity measures. As will become clear, these two measures may give entirely different solutions to the question which of a set of tournaments is most circular.

Suppose the following tournament T_1 contains the results of the pairwise comparisons between alternatives a, b, c, d and e.

T_1	a	b	c	d	e
a	-	1	1	1	0
b	0	-	1	1	1
c	0	0	-	1	1
d	0	0	0	-	1
e	1	0	0	0	-

with graph



The tournament is no linear order: the number of 3-cycles is 3. These 3-cycles $aT_1bT_1eT_1a$, $aT_1cT_1eT_1a$ and $aT_1dT_1eT_1a$, we regard as inconsistencies.

In case of preferences, there are several explanations for the occurrence of inconsistencies, see section 5.2. But, since often we are interested in establishing a transitive tournament, we may ask ourselves, to what extent this tournament does represent an order. And if we draw the conclusion that there must be an underlying order that generated this tournament, one may ask: which order?

In the example above, the number of 3-cycles is 3. Note that they are caused by one pair: eT_1a. Since a linear order is free of cycles, one may say that there are 3 violations of linearity. Therefore we introduce the circularity measure λ, which counts the number of 3-cycles. It was first used by Kendall and Babington Smith (1940). In our example, $\lambda(T_1) = 3$. Intuitively, the larger $\lambda(T)$, the less T represents an order.

Definition 5.2

Let T be a tournament. Then $\lambda(T)$ is the number of 3-cycles.

For all tournaments T, the circularity $\lambda(T)$ is easy to compute:

PROPOSITION 5.1 (Kendall, Babington Smith (1940)). *Let (T,X) be a tournament. Then*

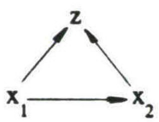
$$\lambda(T,X) = \binom{n}{3} - \sum_{x\in X}\binom{s_x}{2},$$

where $(s_x)_{x\in X}$ is the score vector of (T,X), $n = |X|$.
Moreover, the maximum number, λ_{max}, is $(n^3-n)/24$ if n is odd, $(n^3-4n)/24$ if n is even.

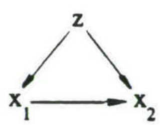
Proof The total number of triads is $\binom{n}{3}$. Given an element x, there are $\binom{s_x}{2}$ triads in which x dominates the other two, meaning that these are non-circular triads. For the determination of λ_{max}, see Moon (1968, page 9). ■

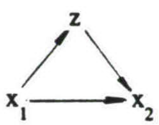
We assume that if $\lambda(T)$ is small, it is compatible with an order, not withstanding a lack of perfection. But which order has to be taken? We propose the weak order $f_{out}(T)$. The motivation for this choice is given in section 5.3.5.

In Kadane (1966), it is proved that $\lambda(T)$ is the number of preference reversals necessary to break all ties in the score vector. To see this, suppose that x_1Tx_2 and x_1 and x_2 do have the same score, say a. Let us consider the effect of reversing the preference between x_1 and x_2. The only triads affected are those containing x_1 and x_2. There are four possible types of such triads:

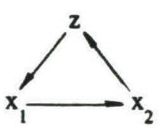


say u in number,





which must number a - 1 - u to assure that the score of x_1 equals a,



which must number a - u to assure that the score of x_2 also equals a.

When x_1Tx_2 is reversed, the first two remain noncircular. The third becomes circular, the fourth ceases to be so. Thus λ has been reduced by 1. (See David, 1988).

A second approach for the determination of circularity is provided by Slater (1961), who searched for the nearest linear orders. To introduce this method, we suppose that we have a linear order in mind. What is its difference or distance to T? To compute this, we define a metric on relations, which are assumed to be subsets of the cartesian product X×X, where X is the space of alternatives.

This metric counts the number of elements of X×X that are in p and not in q or in q and not in p. More popular, it counts the number of ordered pairs upon which p and q do not agree. For example, if $p = a \gg b \gg c$ and $q = b \gg a \gg c$, then d(p,q) = 2: see definition 5.3.

Definition 5.3

Let $p, q \subseteq X \times X$ be two relations on X. The distance between p and q is

$$d(p,q) = |(\hat{p} \cup \hat{q}) - (\hat{p} \cap \hat{q})|,$$

where for arbitrary relations p, $\hat{p} = p \cup id_X$. ($\hat{p}$ is reflexive).

To give an example, let $p = a \gg b \gg c \gg d \gg e$, and let $q = T_1$ be the tournament of page 8. It is easy to verify that d(p,q) = 2, because of $(e,a) \in T_1$, $(e,a) \notin p$ and $(a,e) \in p$, $(a,e) \notin T_1$.

Which linear order is closest to T? Slater (1961) proposed to take the linear orders which minimize the metric d. The corresponding circularity measure *i* is defined by:

Definition 5.4

Let (T,X) be a tournament. Then $i(T,X) = \frac{1}{2} \min_{(L,X) \in \mathbb{L}(X)} d((L,X),(T,X))$ *which is equal to the minimal number of preference reversals needed to convert (T,X) into a linear order.*

In our example, $i(T_1) = 1$. A linear order that has minimal distance to T is called a nearest adjoining order. In our example T_1, there is just one nearest adjoining order: $a \gg b \gg c \gg d \gg e$.

How can we compute *i*(T)? Interesting methods for the computation of Slater's *i* and all nearest adjoining orders have been developed in Remage and Thompson (1966) and Phillips (1967, 1969). The computations require exponential computer time. Another method is described in Bezembinder (1981) (for small values of *i* only).

As proved by Kadane (1966), $\lambda(T)$ is the number of preference reversals necessary to break all ties in the score vector. Hence for all tournaments T: $i(T) \leq \lambda(T)$. It also shows that i and λ belong to a same category: pairwise comparisons are reversed so that the tournament becomes non-circular.

Yet, the circularity measures λ and i may be quite different. For example, take a linear order on n alternatives. Then reverse the comparison between the top and bottom element. In that case, the number of 3-cycles is equal to n - 2, all cycles going through the same arc. As for the measure i, a reversal between top and bottom has the same weight as a reversal between the k-th and the (k + 2)-th alternative, $1 \leq k \leq n - 2$. Hence i is equal to 1. To summarize, λ does not care about counting things double, while i does not take into account the position of the two alternatives in the linear order.

To show another difference, consider the following tournaments.

Let T be

	a	b	c	d	e	f
a	-	1	1	1	0	1
b	0	-	1	1	1	1
c	0	0	-	1	1	1
d	0	0	0	-	1	1
e	1	0	0	0	-	1
f	0	0	0	0	0	-

The unique nearest adjoining order is {a} » {b} » {c} » {d} » {e} » {f}. Hence $i(T) = 1$. By using the formula from proposition 5.1, we calculate $\lambda(T) = 3$.

Now, let S be equal to

	a	b	c	d	e	f
a	-	1	0	1	1	1
b	0	-	1	1	1	1
c	1	0	-	1	1	1
d	0	0	0	-	1	0
e	0	0	0	0	-	1
f	0	0	0	1	0	-

A nearest adjoining ordering is {a} » {b} » {c} » {d} » {e} » {f}, giving $i(S) = 2$. Since there are two 3-cycles, $\lambda(S) = 2$.
We deduce that using i, S is more circular than T, while using λ, T appears to be more circular than S.
Thus, in the determination which of a set of tournaments is most circular, it matters which measure is taken. In the following subsection, we will show that there is no circularity measure that combines all attractive features (to be introduced hereafter) of both i and λ.

5.3.2 An impossibility theorem

In this subsection, we introduce a few desirable properties for circularity measures. We will show that, taken together, they lead to an impossibility.
All these properties are satisfied by either i or λ.

- **Standardization**

  A circularity measure γ on $\mathbb{T}(X)$ satisfies standardization, if for all T: $\gamma(T) \geq 0$; $\gamma(T) = 0$ iff $T \in \mathbb{L}(X)$.

It is clear that the standardization is a desirable property.
In what follows, we frequently need the notion of a tournament with one reversal. If xTy then yxT denotes the tournament T with only (x,y) reversed. The tournament yxT is obtained from T by means of **an elementary change**. To be more precise:

Definition 5.5
Let T be a tournament on X and $x \neq y \in X$. Then yxT is a tournament that is defined by

$$yxT = \{(y,x)\} \cup T\backslash\{(x,y)\}.$$

Looking at the introduction of the measure i in the previous section, we observe that it satisfies the following property:

- **Uniform weighting of elementary changes**

  A circularity measure γ on $\mathbb{T}(X)$ uniformly weighs elementary changes, if for all $x \neq y$, $x' \neq y'$, which are element of X,

  if $\gamma(yxT) > \gamma(T)$ and $\gamma(y'x'T) > \gamma(T)$ then
  $\gamma(yxT) = \gamma(y'x'T)$.

As made clear with the example of the linear order with top and bottom element reversed, λ does not satisfy this property.

A tournament is a linear order if and only if there are no ties in the score vector. Now, $\lambda(T)$ may be reduced by breaking ties in the score vector. Therefore we introduce the following property:

- **Tie-breaking reduction**

  A circularity measure γ on $\mathbf{T}(X)$ satisfies tie-breaking reduction, if for all $x, y \in X$ such that $x \neq y$,

  if $s_x = s_y$ and xTy then $\gamma(yxT) < \gamma(T)$.

The reader will convince himself that Slater's i does not satisfy the tie-breaking reduction property.

Of course, we do not want our circularity measure to depend upon different ways of naming the alternatives. Therefore, we demand

- **Independence of labeling**

  A circularity measure γ on $\mathbf{T}(X)$ is independent of labeling if for all permutations σ of X, $\gamma(\sigma T)) = \gamma(T)$.

All of these four properties are more or less natural. But, as will be proved in theorem 5.2, there is no circularity measure that fulfills all these conditions.

THEOREM 5.2 *Let $|X| \geq 4$. There does not exist a circularity measure γ on $\mathbf{T}(X)$ that satisfies standardization, uniform weighting of elementary changes, tie-breaking reduction and independency of labeling.*

Proof

$$0 = \gamma(T^1) = \gamma\left(\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr] & b \arrow[d] \arrow[dl] \\ d & c \arrow[l] \end{tikzcd}\right) \quad \text{and} \quad \alpha = \gamma(T^2) = \gamma\left(\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr] & b \arrow[d] \\ d \arrow[ur] & c \arrow[l] \end{tikzcd}\right) > 0$$

because of the standardization property.

$$\alpha = \gamma(T^3) = \gamma\left(\begin{tikzcd} a \arrow[r] \arrow[dr] & b \arrow[d] \arrow[dl] \\ d \arrow[u] & c \arrow[l] \end{tikzcd}\right),$$ because of the uniform weighting of elementary changes.

In the last tournament T^3, we have $s_a = s_b$, while aT^3b. Hence, using the tie-breaking reduction property, we conclude that

$$\alpha > \gamma(T^4) = \gamma\left(\begin{tikzcd} a \arrow[dr] & b \arrow[l] \arrow[d] \arrow[dl] \\ d \arrow[u] & c \arrow[l] \end{tikzcd}\right).$$

But this is in contradiction with the independence of labeling, because there exists a permutation σ, such that $\sigma T^4 = T^2$. ■

Because of the differences described in section 5.3.1. and the impossibility theorem presented above, we have to consider the question which measure has to be taken in different situations. In the following subsections, we give a characterization for both measures. Moreover, in section 5.6, we compare them using statistical techniques. These considerations may help us to choose the appropriate measure for a given situation. We conclude our comparison with a discussion in section 5.3.5.

5.3.3 An axiomatization of Slater's i

Take a tournament T and suppose $i(T) = i_0$. Let S be a tournament that differs from T in just one pair. We say that Slater's *i* gives equal weight to all preferences, because $i(S) \in \{i_0 - 1, i_0, i_0 + 1\}$, irrespective of the pair in which the tournaments differ. Apparently, there does not exist a context so that some preference reversals are more sweeping and therefore must receive more weight. To express this more formally, we hereafter introduce a property which we call equability. Together with two mild assumptions, it completely characterizes multiples of Slater's *i*.

We impose the following conditions or axioms upon γ.

A circularity measure γ on $\mathbb{T}(X)$ satisfies

- **Standardization**

  if for all T: $\gamma(T) \geq 0$ and $\gamma(T) = 0$ iff $T \in \mathbb{L}(X)$.

- **Reducibility by elementary changes**

  if for all $T \notin \mathbb{L}(X)$ *there exists a pair* (a,b) *such that* $\gamma(baT) < \gamma(T)$.

- **Equability**

  if for all $x,x',y,y' \in X$ *and* $T,T' \in \mathbb{T}(X)$:

  if xTy, $x'T'y'$, $\gamma(yxT) \neq \gamma(T)$ *and* $\gamma(y'x'T') \neq \gamma(T')$,

  then $|\gamma(yxT) - \gamma(T)| = |\gamma(y'x'T') - \gamma(T')|$.

Note that the equability implies uniform weighting of elementary changes.

The existence of a positive response by means of an elementary change is guaranteed by the reducibility property. The equability means that the effect of all elementary changes at the local level is weighted uniformly. All observed binary decisions are assumed to be of equal importance and are given equal weight. In theorem 5.3 we prove that γ fulfills standardization, reducibility and equability, if and only if there exists a positive number k, such that $\gamma = ki$.

THEOREM 5.3 (Characterization of Slater's *i*) *Let* $X = \{x_1,...,x_n\}$ *be a finite set of alternatives. A circularity measure* $\gamma : \mathbb{T}(X) \to \mathbb{R}$ *satisfies the conditions of standardization, reducibility and equability,*

if and only if there exists a positive number k, such that $\gamma = ki$, *where i is Slater's i.*

Proof (if) To prove the standardization property, we observe that $i(L) = 0$ if $L \in \mathbb{L}(X)$. If $T \notin \mathbb{L}(X)$, then $i(T) = t > 0$. Furthermore, there is a pair xTy, such that $i(yxT) = t - 1$, because i is the minimal number of upsets necessary to reach $\mathbb{L}(X)$. This proves the reducibility. The equability is evident from the definition of i and k.

(only if) Take $T' \notin \mathbb{L}(X)$. Because of the reducibility, there exists a pair x',y' such that $\gamma(yxT') \neq \gamma(T')$. Define $b = |\gamma(y'x'T') - \gamma(T')| > 0$. Then, using the equability, this number b is the same for all T and all pairs x,y such that $\gamma(yxT) \neq \gamma(T)$.

Now consider $\frac{1}{b}\gamma$. Then

$\frac{1}{b}\gamma(L) = 0$ if $L \in \mathbf{L}(X)$ and

$\frac{1}{b}\gamma(yxT) - \frac{1}{b}\gamma(T) = 0$ or ± 1 for all T and all x,y.

We prove that $\frac{1}{b}\gamma = i$, which means that we may take k = b.
Take a tournament T. If $T \in \mathbf{L}(X)$ then $\frac{1}{b}\gamma(T) = i(T) = 0$. Now suppose $T \notin \mathbf{L}(X)$. Using the standardization and reducibility properties and the fact that $|\mathbf{T}(X)| < \infty$, we may deduce that there exists a sequence of tournaments

$$
\begin{array}{ll}
T & \notin \mathbf{L}(X) \\
y^1x^1T & \notin \mathbf{L}(X) \\
y^2x^2(y^1x^1T) & \notin \mathbf{L}(X) \\
.. & \notin \mathbf{L}(X) \\
.. & \notin \mathbf{L}(X) \\
y^sx^s(y^{s-1}x^{s-1}(.....T)) & \in \mathbf{L}(X)
\end{array}
$$

such that the corresponding values of $\frac{1}{b}\gamma$ are strictly descending. But we already proved that the drop in value must be in steps of 1, which means that $\frac{1}{b}\gamma(T) = s \geq i(T)$. Now, take any sequence of tournaments, starting with T and ending up in $\mathbf{L}(X)$. Again, changes in value of $\frac{1}{b}\gamma$ are restricted to -1, 0 and 1. Because $\frac{1}{b}\gamma(L) = 0$ if $L \in \mathbf{L}(X)$ and $\frac{1}{b}\gamma(T) = s$, this sequence (without T) contains at least s tournaments. Thus $i(T) \geq s$, implying that $i(T) = s = \frac{1}{b}\gamma(T)$. ∎

Theorem 5.3 means that the set of circularity indices that satisfy the three conditions, is parameterized by the set of positive numbers, or similarity transformations, which means that for each X, such a circularity measure is on a ratio scale. The value of k(X) may be computed by taking a circular tournament T': $k(X) = \gamma(T',X)/i(T',X)$.

The independence of the three conditions of theorem 5.3 if $|X| \geq 3$.
(*A*) Take $X = \{a_1,a_2,a_3,a_4\}$. If $T^1 = a_1 \gg a_2 \gg a_3 \gg a_4$, then we define $\gamma(T^1) = 1$, for all other linear tournaments L we define $\gamma(L) = 0$. Next, let T^2 be equal to

Define $\gamma(T^2) = 2$. For all other tournaments T', we define $\gamma(T') = i(T')$. This circularity measure is not independent of labeling. But it satisfies the reduction property and equability of elementary changes. To show the reduction property: All tournaments $T' \neq T^2$ can be converted into a linear tournament $L' \neq T^1$, by means of an elementary change. As for the equability: All tournaments $T' \neq T^2$ are assigned a value 0 or 1. There is no tournament T' with $\gamma(T') = 0$ that can be converted into T^2 by means of an elementary change. It is clear that this choice of γ does not satisfy standardization.

If $|X| = 3$ then take $\gamma \equiv i + 1$. Then γ does not fulfill standardization.

(*B*) Take $\gamma \equiv \lambda$, where $\lambda(T)$ is the number of 3-cycles of T. It does not satisfy the equability if $|X| \geq 4$: reverse top and bottom element in a linear order. To show that it satisfies reducibility, find that alternative j with highest score s_j, that is dominated by an alternative k with lowest score s_k. Take (a,b) = (k,j). As for the case $|X| = 3$, give different values to the two possible 3-cycles.

(*C*) Take γ as follows. If $T \in \mathbb{L}$, then $\gamma(T) = 0$, otherwise $\gamma(T) = 1$. This measure does not satisfy the reducibility condition.

From part A of the verification of the independence of the conditions, one may conjecture the following corollary. Its proof is completely analogous to that of theorem 5.3.

COROLLARY 5.4 *Let $X = \{x_1,\ldots,x_n\}$ be a finite set of alternatives. A circularity measure $\gamma : \mathbb{T}(X) \to \mathbb{R}$ satisfies independence of labeling, reducibility and is equable,*
if and only if there exists a positive number k and a number c, such that $\gamma = ki + c$, where i is Slater's i. ■

The set of circularity measures that satisfy these conditions are on an interval scale.

In preparation of proposition 5.5, we introduce the following conditions, that may be seen as weaker versions of the equability condition.

- **The smallest step property**

  A circularity measure γ on $\mathbb{T}(X)$ satisfies the smallest step property, if for all T, the following holds. If aTb then there are no T' with $\gamma(T')$ strictly between $\gamma(T)$ and $\gamma(baT)$.

- **Level dependence**

  A circularity measure γ on $\mathbb{T}(X)$ is level dependent if there exist functions $v,w : \mathbb{R} \to [0,\infty)$, such that for all $x,y \in X$ and all T:

  $$\text{if } \gamma(yxT) \leq \gamma(T) \text{ then } \gamma(yxT) = v(\gamma(T)), \text{ else } \gamma(yxT) = w(\gamma(T)).$$

Note that if a circularity measure is level dependent, then it uniformly weighs elementary changes (introduced in section 5.3.2.).

PROPOSITION 5.5 *Let X be a finite set of alternatives.*

(a) A circularity measure γ on $\mathbb{T}(X)$ satisfies standardization, reducibility and the smallest step property, if and only if there exists a strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$, such that $\gamma = \phi \circ i$, where i is Slater's i.

(b) A circularity measure γ on $\mathbb{T}(X)$ satisfies standardization, reducibility and is level dependent, if and only if there exists a strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$, such that $\gamma = \phi \circ i$, where i is Slater's i.

Proof We only give a brief sketch of the proof of the 'only if' of part a, the rest is left to the reader. We may define sets or levels of tournaments, where each level consists of tournaments that need the same number of preference reversals to convert a tournament T into a linear tournament, under the restriction that the corresponding sequence of circularities is strictly decreasing. Now, define the function ϕ for each level. ■

Such measures are on a restricted ordinal scale. It is restricted, because, using the standardization property, we must have $\phi(0) = 0$, which is the natural zero point of circularity. Ordinal scales with a zero point are rare. To the best of our knowledge, the only two places in the literature where they do come up are in Bezembinder and van Acker (1987, page 102) and

Coombs and Avrunin (1988, page 82).

Since the positive number k, defined in the proof of theorem 5.3, depends upon the choice of X, we cannot state:

for all $(T,X) \in \mathbf{T}(X)$ and all $(S,Y) \in \mathbf{T}(Y)$:

$\gamma(T,X) > \gamma(S,Y)$ iff $i(T,X) > i(S,Y)$.

This may be seen by taking $\gamma = |X|i$, satisfying all three conditions of theorem 5.3, and choosing (T,X) and (S,Y) such that $i(T,X) = i(S,Y)$, while $|X| > |Y|$. Indeed, we have $\gamma(T,X) > \gamma(S,Y)$.

To achieve a result that is independent of the set of alternatives X, we introduce an additional axiom for a circularity measure on $\mathbf{T}^*$. We let $\mathbf{T}^*$ be the set of tournaments on arbitrary finite $X = \{x_1,...,x_k\}$. Also see chapter 3, section 3.5.4.

- **Extensiveness**

  A circularity measure γ on $\mathbf{T}^*$ satisfies extensiveness, if for all tournaments (T,Y) and (T,Z) with $Y \cap Z = \emptyset$:

  $$\gamma((T,Y) \gg (T,Z)) = \gamma(T,Y) + \gamma(T,Z).$$

As for the name, attributes that have additive properties have traditionally been called extensive in the theory of measurement (see Roberts, 1979).

We have the following theorem. Its proof is analogous to the proof of theorem 5.3.

THEOREM 5.6 *A circularity measure γ: $\mathbf{T}^* \to \mathbb{R}$ satisfies standardization, reducibility and equability of elementary changes for each finite set X, and satisfies the extensiveness property if and only if there exists a positive number k, such that $\gamma = ki$, where i is Slater's i.* ■

5.3.4 An axiomatization of Kendall-Babington Smith's lambda

In the previous subsection, we characterized Slater's *i*. The distinguishing property was the equability. As already made clear in section 5.3.1, the number of 3-cycles is not equable. A reversal between top and bottom element of a linear order introduces more 3-cycles than other reversals. Yet, it satisfies a related property, which we call outscore equability:

Two elementary changes are weighted uniformly, if the domination difference between the elements of both pairs is the same, more precise:

Let $s = (s_x)_{x \in X}$, be the score vector of T.

A circularity measure γ on $\mathbb{T}(X)$ satisfies

- **Standardization**

    if for all T: $\gamma(T) \geq 0$; $\gamma(T) = 0$ iff $T \in \mathbb{L}(X)$.

- **Outscore equability**

    if for all $x,y,x',y' \in X$ and $T,T' \in \mathbb{T}(X)$:
    if xTy and $x'T'y'$ and $s_x - s_y = s'_{x'} - s'_{y'}$,
    then $\gamma(yxT) - \gamma(T) = \gamma(y'x'T') - \gamma(T')$.

One may view the score of an alternative as a measure of the strength or desirability of that alternative, and thus order the alternatives by their score, as is done in ranking players in a win or loose round robin tournament. The difference between the scores may be seen as the distance on that scale given by the scores. The outscore equability relates the change of circularity that a reversal may cause, to that difference. In theorem 5.10 we prove that γ satisfies the standardization and outscore equability property, if and only if $\gamma = k\lambda$, for some $k > 0$, where $\lambda(T)$ is the number of 3-cycles of T.

LEMMA 5.7 *The circularity measure λ satisfies standardization and outscore equability.*

Proof The standardization is evident. As for the outscore equability, suppose xTy. Now consider $\lambda(yxT) - \lambda(T)$. It is easy to verify that this is equal to

$$\binom{s_x}{2} + \binom{s_y}{2} - \binom{s_x - 1}{2} - \binom{s_y + 1}{2}$$

which is equal to $s_x - s_y - 1$ if $s_x \geq 1$ and $s_y \geq 0$. ∎

Note that Slater's *i* does not satisfy the outscore equability. To show this, recall that the number of 3-cycles of a tournament is equal to the number of preference reversals necessary to break all ties in the score vector (Kadane, 1966). Sometimes, such a preference reversal reduces the value of i, sometimes it does not, showing that Slater's i is not outscore equable.

In preparation of theorem 5.10, we present the following two lemmas. The first lemma shows that the larger the difference between the scores of x and y, the larger is the effect of the preference reversal between x and y.

LEMMA 5.8 *Suppose that a circularity measure γ on $\mathbf{T}(X)$ satisfies standardization and outscore equability.*
Then, for all tournaments $T \in \mathbf{T}(X)$ and all $x \neq y \in X$ such that xTy, we have $\gamma(yxT) - \gamma(T) = a(s_x - s_y - 1)$ for some constant $a > 0$.

Proof From the outscore equability, we deduce that if xTy then $\gamma(yxT) - \gamma(T) = f(s_x - s_y)$, for some $f : \mathbf{Z} \to \mathbf{R}$. We prove that there are real numbers a and b, such that for all $t \in \{1,2,...,|X|-1\}$, $f(t) = at + b$. Take a tournament $T \in \mathbf{L}(X)$, $|X| = n \geq 3$, and $x,y,z \in X$, such that $s_z = 0$, $s_y = 1$, $s_x \geq 2$ and xTyTz. Because $yx(zyT) = zy(yxT)$, we obtain the relation
$f(1) + f(s_x) = f(s_x - 1) + f(2)$. This may be rewritten as

$$f(s_x) - f(s_x-1) = f(2) - f(1).$$

Because s_x can take any value between 2 and n-1, this implies that $f(t) = at + b$, $t \in \{1,2,3,...,n-1.\}$. If $a = 0$ then $f(t) = b$. Using the standardization, we see that $b > 0$. But then we arrive at a contradiction. For take T to be linear and x and y are two consecutive alternatives in that linear ordering. Then yxT is a linear ordering too, which means that $0 = \gamma(yxT) = b$. Hence $a \neq 0$. If $T \in \mathbf{L}(X)$ and $s_x - s_y = 1$, then $yxT \in \mathbf{L}(X)$, thus $f(1) = 0$. But $f(1) = a + b$, hence $b = -a$. Now, $f(t) = a(t-1)$. Because $\gamma \geq 0$, we have $f \geq 0$, thus $a > 0$.
Now, consider the case $s_x - s_y \leq 0$ and xTy. Then we have $s'_y \geq s'_x + 1$ and yT'x, T' = yxT, where s' stands for the score in yxT. Applying what already has been proved, we obtain $\gamma(T) = \gamma(yxT) + a(s'_y - s'_x - 1)$. Because $s'_y = s_y + 1$ and $s'_x = s_x - 1$, we finally arrive at
$\gamma(yxT) = \gamma(T) + a(s_x - s_y - 1)$. ■

LEMMA 5.9 *Let $\phi : \mathbb{N} \cup \{0\} \to \mathbb{R}$. Then $\phi \circ \lambda$ satisfies standardization and outscore equability, iff $\phi(t) = kt$, for some $k > 0$.*

Proof (only if) Because of lemma 5.8, $\phi\circ\lambda(yxT) - \phi\circ\lambda(T) = a(s_x - s_y - 1)$ for some number $a > 0$. We prove that $\phi(t) = at$. We use induction to $K = \lambda(T)$.
(Basis step) If $K = 0$, then $T \in \mathbb{L}(X)$. So, using the standardization property, we obtain $0 = \phi\circ\lambda(T) = \phi(0)$.
(Induction step). Suppose that for $k \leq K$, we have $\phi(k) = ak$. Now take T such that $\beta = \lambda(yxT) = K+1$ and $\alpha = \lambda(T) \leq K$. Then

$$\phi(\beta) - \phi(\alpha) = \phi\circ\lambda(yxT) - \phi\circ\lambda(T) = a(s_x - s_y - 1). \qquad \text{(lemma 5.8)}$$

Thus

$$\begin{aligned} \phi(\beta) &= \phi(\alpha) + a(s_x - s_y - 1) \\ &= a\lambda(T) + a(s_x - s_y - 1). && \text{(induction hypothesis)} \end{aligned}$$

Hence

$$\begin{aligned} \phi(\beta) &= a(\lambda(T) + s_x - s_y - 1) \\ &= a(\lambda(T) + \lambda(yxT) - \lambda(T)) && \text{(lemma 5.7)} \\ &= a\lambda(yxT) = a\beta. \end{aligned}$$

(if) The standardization is obvious. The measure γ is outscore equable, because $k\lambda(yxT) = k[\lambda(T) + s_x - s_y - 1] = k\lambda(T) + k(s_x - s_y - 1)$.■

THEOREM 5.10 (Characterization of λ) *The circularity measure γ on $\mathbb{T}(X)$ satisfies standardization and is outscore equable iff there exists a number $k > 0$ such that $\gamma = k\lambda$.*

Proof (if) This is already done in lemma 5.9.
(only if) The measure γ is completely determined by the choice of the number a in lemma 5.8, for example λ is determined by $a = 1$. Take a tournament T. It may be obtained through a sequence of reversals between pairs of alternatives, starting with a linear tournament L. Suppose we have the chain

$$L = T^0 \to T^1 \to T^2 \to \dots \to T^n = T,$$

where two consecutive tournaments only differ in one pair: $T^i = y_i x_i T^{i-1}$, $x_i T^{i-1} y_i$, $i \in \{1,\dots,n\}$. Next, we define
$d_i = s(x_i,T^{i-1}) - s(y_i,T^{i-1})$, where $s(x,T^i)$ is the score of x in T^i. From

lemma 5.8, we may deduce: $\gamma(T^1) = a(d_1 - 1)$, $\gamma(T^2) = a(d_1 - 1) + a(d_2 - 1)$, ..., $\gamma(T) = \sum_{i=1}^{n} a(d_i - 1)$. But this is equal to $a \sum_{i=1}^{n} (d_i - 1) = a\lambda(T)$. ■

Note that lemma 5.9 cannot replace theorem 5.10. For, suppose we only have lemma 5.9. Then we cannot be sure that there does not exist a circularity measure γ, satisfying standardization and the outscore equability, while $\gamma \neq \phi \circ \lambda$ for all choices of ϕ. For example, it could be possible that for two tournaments T and S we have $\lambda(T) = \lambda(S)$, while $\gamma(T) \neq \gamma(S)$, meaning that the axioms would not imply regularity of the scale, see Roberts (1979, theorem 2.1). Also consider the discussion preceding theorem 5.6.

Logical independence of the conditions of theorem 5.10.

(1) The measure i satisfies standardization. It is not outscore equable.

(2) If only the outscore equability must be satisfied, we may take $\lambda + 1$.

5.3.5 Discussion: comparison and applicability

As is made clear in the characterizations of Slater's i and λ, again we meet two levels of analysis. First of all, we have Slater's i which weights all preferences uniformly, because for all tournaments T and all $x,y \in X$, if $i(T) = i_0$ then $i(yxT,X) \in \{i_0 - 1, i_0, i_0 + 1\}$. This matches a local level of analysis, where all preferences receive equal weight. There is no information, order or context available causing some inconsistencies to be more sweeping and therefore deserving more weight. Secondly, we have Kendall-Babington Smith's lambda. The characterization of λ shows that changes in the existing status quo, are taken into account by using scores. One may assert that the status quo is a weak order, determined by the scores: the larger the difference between the scores of x and y, the more impact the reversal between x and y has on the circularity. Hence, although in general the tournament T is no weak order, it bears in it a weak order, for example the one that may be derived by looking at the scores. This fits a more global level of analysis, which consists of balancing and weighting pairwise comparisons. Also see section 5.6.1.

Therefore, we discern two domains of applicability for our circularity measures. Slater's i may be useful when there is no reason to believe that the comparisons satisfy the conditions of a linear order. This is the case when for example pairwise comparisons are generated randomly or when we have a problem where we may expect a large number of inconsistencies. Then Slater's i indicates to what extent these pairwise comparisons constitute an order. In all other cases, when we have to decide to what extent an order is disturbed, we recommend the use of λ.

5.4 CIRCULARITIES DEFINED BY THE SCORE VECTOR

As stated in proposition 5.1, the number of 3-cycles in a tournament (T,X) is $\lambda(T,X) = \binom{n}{3} - \sum_{x \in X} \binom{s_x}{2}$, *where* s_x *is the score of alternative x. Thus* λ *is a function of the score vector. In this section, we characterize the circularity measures that are a function of the score vector, by using Independence of 3-Cycle Orientation. Furthermore, we show that the strict utility model may be seen as a probabilistic analogon of this notion.*

5.4.1 Independence of 3-Cycle Orientation

First, we introduce rT. Let $T \in \mathcal{T}(X)$. Suppose r is a 3-cycle in T. Then rT is the tournament which can be obtained from T by reversing (the orientation of) the 3-cycle r in T. If the 3-cycle r is equal to aTbTcTa, we may denote this tournament with <abc>T.

If alternatives form a 3-cycle, we may interpret this as an indifference of these alternatives. Therefore, we demand that the orientation of a 3-cycle does not have consequences for the circularity:

- **Independence of 3-cycle orientation**

  A circularity measure γ *on* $\mathcal{T}(X)$ *is independent of 3-cycle orientaion, if for all* $T \in \mathcal{T}(X)$ *and all 3-cycles r in T:*

  $$\gamma(T) = \gamma(rT).$$

The effect of reversals of 3-cycles on rankings have been studied extensively: see for example Moon (1968, page 73), Henriet (1985), Bouyssou (1992) and the chapters 2 and 3 of this monograph.

The following theorem provides a characterization of these measures: they are functionally dependent of the score vector.

THEOREM 5.11 *Let $X = \{x_1,...,x_n\}$ be a finite set of alternatives. A circularity measure γ on $\mathbb{T}(X)$ is independent of 3-cycle orientation iff there exists a function $f : \mathbb{R}^n \to \mathbb{R}$, $n = |X|$, such that $\gamma(T) = f(s_1,...,s_n)$, where $(s_1,...,s_n)$ is the score vector of T.*

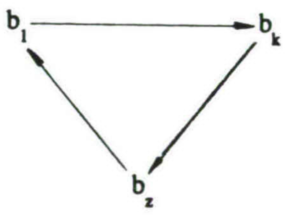
Proof (if) Because f only depends on the score vector, which is the same for T and rT for all 3-cycles r of T, γ is independent of 3-cycle orientation.

(only if) Let X be equal to $\{a_1,...,a_n\}$. Suppose that the score vectors of T and S are the same: $(s_1,...,s_n)$, where s_i is the score of alternative a_i, $i \in \{1,...,n\}$. We must show that $\gamma(T) = \gamma(S)$ if γ is independent of 3-cycle orientation. If we prove that S may be obtained from T through a series of 3-cycle reversals, we are finished, because of the independence condition. Just for the convenience, we assume that $\{b_1,...,b_n\}$ is a labeling of the set X such that the corresponding scores are not increasing.

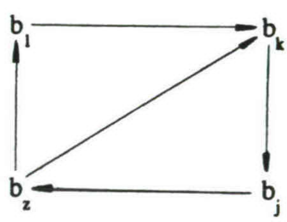
Take the alternative b_1. Consider the smallest k such that $(T,\{b_1,b_k\}) \neq (S,\{b_1,b_k\})$. Suppose b_1Tb_k, which means that b_kSb_1. Because $s_{b_1}(T) = s_{b_1}(S)$, there exists an alternative b_z with b_1Sb_z and b_zTb_1. By the minimality of k, $z > k$, thus $s_k \geq s_z$.

We consider two cases. Case I: b_kTb_z, Case II: b_zTb_k.

In case I, we have the 3-cycle $b_1Tb_kTb_zTb_1$. See figure 5.2. This 3-cycle does not contain arcs between b_1 and b_p with $p < k$. Now, $T' = \langle b_1,b_k,b_z\rangle T$ is a tournament that is obtained from T by a 3-cycle reversal. Moreover, the pairwise comparisons between b_1 and b_i, $i \in \{1,...,k\}$ is the same both for S and $\langle b_1,b_k,b_z\rangle T$.



Case I



Case II

Figure 5.2

Now consider case II. Because $s_k \geq s_z$, we know that there exists an alternative b_j such that b_kTb_j, b_jTb_z. Thus we have the 4-cycle

$b_1 T b_k T b_j T b_z T b_1$. See figure 5.2. Define $r = b_z T b_k T b_j T b_z$. Then we have the following 3-cycle in $T' = \langle b_z, b_k, b_j \rangle T$: $v = b_1 T' b_k T' b_z T' b_1$ and $\langle b_1, b_k, b_z \rangle \langle b_z, b_k, b_j \rangle T$ is a tournament that is obtained from T through a series of 3-cycle reversals. Moreover, in $\langle b_1, b_k, b_z \rangle \langle b_z, b_k, b_j \rangle T$ the relation between b_1 and b_i, $i \in \{1,...,k\}$ is the same as in S.
Repeat the process described above, until the arc between b_1 and alternative c is the same in T and S for all c. After this operation, we 'remove' b_1 and all its arcs from both tournaments, which result in two tournaments on the alternatives $\{b_2,...,b_n\}$ with equal score vector, because 3-cycle reversals do not have impact on the score vector. Continue as before. ■

From theorem 5.11 we deduce:

COROLLARY 5.12 *If γ is independent of 3-cycle orientation, then γ is independent of n-cycle orientation, $n \in \{3,4,5,...\}$.*

Proof If γ is independent of 3-cycle orientations, we know from theorem 5.11 that for all T, $\gamma(T) = f(s_1,...,s_n)$ for some $f : \mathbb{R}^n \to \mathbb{R}$. Since reversing an n-cycle does not change the score vector, γ is independent of n-cycle orientation. ■

Let $\gamma(T) = \sum_{x=1}^{n} x s_x$. If r is a circuit in T, we see that $\gamma(T) = \gamma(rT)$, hence γ is independent of 3-cycle orientation. But γ is not neutral or independent of labeling: $\gamma(T) \neq \gamma(\sigma T)$ for all permutations σ of X.

If we want our circularity measures to be independent of labeling, we must demand that γ is a function of the score sequence: see theorem 5.13. Let X $= \{a_1,...,a_n\}$ and $T \in \mathbb{T}(X)$. The sequence $(s_1,...,s_n)$ such that
$s_1 \geq s_2 \geq ... \geq s_n$ is the score sequence of T, if there exists a labeling $\{a_1,...,a_n\}$ of X such that for $i \in \{1,...,n\}$, s_i is the score of alternative a_i.

THEOREM 5.13 *Let* $X = \{x_1,...,x_n\}$ *be a finite set of alternatives. A circularity measure* γ *on* $\mathbb{T}(X)$ *is independent of 3-cycle orientation and is independent of labeling iff*
for each finite X *there exists a function* $f : \mathbb{R}^n \to \mathbb{R}$, $n = |X|$, *such that* $\gamma(T) = f(s_1,...,s_n)$, *where* $(s_1,...,s_n)$ *is the score sequence of the tournament* T.

Proof (if) Evident, because for all σ, the score sequences of T and σT are the same. (only if) Suppose that the score sequences $(s_1,...,s_n)$ of T and S are the same. Then there exist σ_T and σ_S such that the score of alternative a_i is equal to s_i, both in $\sigma_T T$ and $\sigma_S S$, $i \in \{1,...,n\}$. Because γ is independent of labeling, $\gamma(\sigma_T T) = \gamma(T)$ and $\gamma(\sigma_S S) = \gamma(S)$. Continue with $\sigma_T T$ and $\sigma_S(S)$ as in the proof of theorem 5.11. ■

It is evident that λ is independent of 3-cycle orientation and is a function of the score sequence.

There are several circularity measures that are not independent of 3-cycle orientation. We investigate the independence for a few circularity measures that we introduce in definition 5.6. For all these circularity measures we have: the higher its value, the more circular is the tournament.

Definition 5.6

Let T be a tournament.

(a) $\chi(T)$ is the number of 4-cycles in T.

(b) $\delta(T)$ is the proportion of the total number of preferences that are part of a circuit in T.

(c) Bezembinder's ρ (Bezembinder, 1981, David, 1988): the dimension of the cycle space, which appears to be equal to $N\delta + k - n$, where k is the number of indifference classes of the transitive closure, $n = |X|$ and $N = n(n-1)/2$. Unlike λ, it takes into account all p-cycles, $p = 3,4,...$ This cycle space is obtained in the following manner. Take a circuit. Assign to every pair of alternatives (x,y) a 1 if xTy or yTx is in this circuit, a 0 otherwise. This gives an N dimensional vector. The dimension of the subspace of all those vectors is ρ. Further $\rho_{max} = 1 - n + \binom{n}{2}$.

(d) $sc(T) = d(T, f_{out}(T))$. This circularity measure determines the distance to the weak order given by the ranking rule f_{out}.

To show the working of these measures, we consider the following tournament.

Let S be

	a	b	c	d	e
a	-	1	1	1	0
b	0	-	1	1	1
c	0	0	-	1	1
d	0	0	0	-	1
e	1	0	0	0	-

Now $\lambda(S) = 3$, with the following 3-cycles: {a,b,e}, {a,c,e} and {a,d,e}. Further, $i(S) = 1$ since the reversal of the relation between the alternatives e and a gives a linear tournament. $\chi(S) = 3$; we observe 3 four cycles, {a,b,c,e}, {a,b,d,e} and {a,c,d,e}. The tournament S is irreducible, because of the 5-cycle {a,b,c,d,e}, hence $\delta(S) = \delta_{max} = 1$. Thus $\varepsilon(S) = 10$, $\rho(S) = 4$.

Verifications of the independence of 3-cycle orientation:

Slater's i is not independent of 3-cycle orientation. To see this, observe that $i(S) = 1$, while $i(\langle ace\rangle S) = 2$. The measure sc, which is defined by $sc(T) = d(T,f_{out}(T))$ is not independent of 3-cycle orientation: $f_{out}(S) = \{a,b\} \gg \{c\} \gg \{d,e\}$. Then $sc(S) = 4$. Take the cycle r = aScSeSa. Then $f_{out}(rS) = \{a,b\} \gg \{c\} \gg \{d,e\}$, while $sc(rS) = 6$. The circularity measure χ is not independent of 3-cycle orientation. Indeed, $\chi(S) = 3$, but $\chi(\langle e,c,d\rangle\langle a,c,e\rangle S) = 2$. Of course, this does not mean that we cannot find a formula for $\chi(T)$: Storcken (private communication) provided the following proposition:

PROPOSITION 5.14 (Storcken) *Let T be a tournament. Let A be the tournament matrix. Then* $\chi(T) = \frac{1}{4}\,trace(A^4)$.

Proof Let $B = A^4$. Then b_{ii} is the number of paths of length 4, starting and ending with alternative i. Since each path contains four different alternatives, each path is counted four times. ■

As for the computation of $\delta(T)$, we mention the following.

Suppose that the transitive closure of T is $X_1 \gg \ldots \gg X_k$. Now take

$x, y \in X$. The relation between x and y lies on a cycle, iff there exists an $i \in \{1,\dots,k\}$ such that $x, y \in X_i$. Thus, if we define $t_i = |X_i|$, we have $\delta(T) = \frac{1}{N} \sum_{i=1}^{k} t_i(t_i-1)/2$, $N = n(n-1)/2$, $n = |X|$.

If two tournaments have the same score sequence, then the transitive closure is the same for both tournaments, see chapter 1, proposition 1.3. Hence δ and ρ are independent of 3-cycle orientation.

5.4.2 The strict utility model: a probabilistic analogon

In this subsection, we give a probabilistic analogon of independence of 3-cycle orientation. We assume that a person does compare all pairs of alternatives at least once, giving rise to a system $P = (p_{ab})$, $a,b \in X$, where p_{ab} is the frequency with which a is preferred to b. Furthermore, we assume that the person is forced to make a choice between each pair. For all $a \neq b$ we have $p_{ab} + p_{ba} = 1$. Such a system is called a forced pair comparison system.

In many cases transitivity is violated, hence all measurement techniques break down, because they use the concept of an order. Nevertheless, in many cases there is a pattern to the inconsistencies in the pairwise comparisons, hence they are called 'variations'. For example, we may have strong stochastic transitivity, see Roberts (1979):

Definition 5.7

A system $P = (p_{ab})$, $a,b \in X$, satisfies Strong Stochastic Transitivity (SST) if for all a,b and c

$$\text{if } p_{ab} \geq 1/2 \;\&\; p_{bc} \geq 1/2 \text{ then } p_{ac} \geq \max\{p_{ab}, p_{bc}\}.$$

Such a system gives rise to (various) so-called weak stochastic orders $(a_1, a_2, \dots, a_n)$ where $p_{ij} \geq 1/2$ if $i < j$.

Or one may have the strict utility model, see Roberts (1979):

Definition 5.8

The forced pair comparison system $P = (p_{ab})$ satisfies the strict utility model if for all $x \in X$ there exist weights $w(x)$ such that for all $a,b \in X$

$$p_{ab} = \frac{w(a)}{w(a) + w(b)}.$$

From this representation, one may deduce that $p_{aa} = 1/2$ for all a, hence $w(a) \neq 0$ for all a, which implies that $p_{ab} \neq 0,1$ for all pairs a,b.
Note that the weights w, which are unique up to a multiplication with a constant $k \in \mathbb{R}\backslash\{0\}$, give a weak order: $a \gg b$ iff $w(a) > w(b)$ iff $p_{ab} > 1/2$, $a \cong b$ iff $w(a) = w(b)$ iff $p_{ab} = 1/2$.

PROPOSITION 5.15 (Roberts 1979). *Suppose the system $P = (p_{ab})$, $a,b \in X$ satisfies the strict utility model. Then it satisfies strong stochastic transitivity.* ■

If a set (p_{ab}), $a,b \in X$ is given, we can easily test whether or not it satisfies the strict utility model, see Roberts (1979). There it is proved that

the system P satisfies the strict utility model iff

$p_{ab}p_{bc}p_{ca} = p_{ac}p_{cb}p_{ba}$, for all a,b and c and $p_{ab} \neq 0,1$ for all a,b.

This means that the two 3-cycles $a \gg b \gg c \gg a$ and $a \ll b \ll c \ll a$ do have the same probability.

There are several other transitivity conditions, see Roberts (1979). A person may be called probabilistically consistent if its pairwise comparisons satisfy such a condition.

In preparation of theorem 5.17, we introduce the (maximum) likelihood.
Let for al $a,b \in X$, p_{ab} be the probability that alternative a beats alternative b. The likelihood of a tournament T therefore is $L(T) = \prod_{(a,b)\in T} p_{ab}$, where we assume that all choices are made independently of one another.

Now, suppose that we have an irreducible tournament T. We could have generated this tournament using probabilities p_{ab}, $a,b \in X$. We may wonder which values of p_{ab} are the most natural. The strategy we follow is: choose those p_{ab} that maximize L, giving rise to the *maximum* likelihood estimates p_{ab} of the 'true' values π_{ab}.

In maximizing the likelihood, we want to restrict the choices of the system $P = (p_{ab})$, $a,b \in X$, to subsets of the whole space of systems P, that we can

interpret or describe in terms of transitivity conditions. For example, we may want to maximize the likelihood under the restriction that the system P satisfies weak stochastic transitivity, see definition 5.11.
In this section we consider systems that satisfy the strict utility model.

If the system $P = (p_{ab})$, $a,b \in X$ satisfies the strict utility model, we may compute the likelihood of a tournament T given this system P:

$$L(T) = \prod_{(a,b)\in T} p_{ab} = \frac{\prod w_a^{s_a}}{\prod_{(a,b)\in T} (w_a + w_b)}, \qquad \text{(a)}$$

where (s_a), $a \in X$ is the score vector.
If the tournament is reducible, it is easy to verify that the maximum value cannot be attained by systems satisfying the strict utility models. For, let (w_a), $a \in X$, be the point where the maximum value is obtained. Then, if $T = (T,Y) \gg (T,Z)$, we replace the values w_z by kw_z, $0 < k < 1$ and $z \in Z$. But then the value of L(T) grows, a contradiction.
In case T is irreducible, there exists a unique point where the maximum value is obtained, see David (1988, page 62) or Moon (1968, page 43). There, one may find an iterative scheme for the determination of these weights:
Let $n = |X|$, $w_i^{(o)} = 1/n$ and

$$w_i^{(k)} = \frac{s_i}{\sum_{j\neq i} (w_i^{(k-1)} + w_j^{(k-1)})^{-1}} \text{ for } i = 1 \text{ to } n. \qquad \text{(b)}$$

If we let $w(a_i)$ be the limit of this series as $k \to \infty$, we define $p_{ab} = w(a)/(w(a)+w(b))$.

Combining the formulas (a) and (b) given above, we see that, in case T is irreducible, the likelihood is a function of the score vector or sequence.
Summarizing, if we restrict the systems P to the strict utility model, the maximum likelihood is a function of the score vector.

To obtain a converse result, we continue with lemma 5.16, where an equivalent form of the strict utility model is presented by

$$p_{ab} = F(\theta_a - \theta_b), \text{ where } F(t) = 1/(1 + e^{-t}).$$

To see the equivalence, define $\theta_i = \ln(w_i)$.

LEMMA 5.16 (Bühlmann and Huber (1963)). *Suppose $p_{ab} \neq 0$ for all pairs a, b. Then the maximum likelihood of a tournament is a function of the score sequence iff $P = (p_{ab})$ is of the form*

$$p_{ab} = F(\theta_a - \theta_b),$$

for real θ_a, $a \in X$ and $F(t) = 1/(1 + e^{-t})$. ∎

The sentence 'the maximum likelihood is a function of the score vector', must be read as follows. If we insist upon the fact that the score vector is a sufficient statistic for the maximum likelihood, we implicitly restrict the domain of possible systems $P = (p_{ab})$. Lemma 5.16 shows that the restriction is to strict utility models.

Now we can state and prove the following theorem, illustrating a probabilistic analogon of independence of 3-cycle orientation.

THEOREM 5.17 *We assume that $p_{ab} \neq 0$ for all $a,b \in X$. Restricted to irreducible tournaments, the following two statements are equivalent.*

(1) The maximum likelihood of a tournament is Independent of 3-Cycle Orientation.

(2) For each (T,X), the maximizing system P satisfies $p_{ab}p_{bc}p_{ca} = p_{ac}p_{cb}p_{ba}$ for all a,b,c and $p_a \neq 0,1$ for all a.

Proof (1 ⇒ 2) Because of theorem 5.11, the maximum likelihood, (which is independent of labeling) is a function of the score vector. As shown in lemma 5.16, $P = (p_{ab})$ satisfies the strict utility model. In Roberts (1979) it is proved that the system satisfies the strict utility model if and only if $p_{ab}p_{bc}p_{ca} = p_{ac}p_{cb}p_{ba}$, for all a,b and c.

(2 ⇒ 1) The system P satisfies the strict utility model, see Roberts (1979). Hence, the maximum likelihood is a function of the score vector, as is described before lemma 5.16. Now, theorem 5.11 implies that it is independent of 3-cycle orientation. ∎

5.5 CHARACTERIZATION OF EPSILON

In section 5.4.1, we introduced the circularity measure δ, which assigns to a tournament T the proportion of all pairwise comparisons that are part of a cycle in T. We consider the measure $\varepsilon = N\delta$, where N is the number of pairwise comparisons. Thus $\varepsilon(T)$ is the total number of pairwise comparisons that are on a cycle in T. In this section, we assume that the circularity measures are defined on $\mathbf{T}$.

Let d be the distance introduced in definition 5.3. For any tournament T, $f_{trans}(T)$ is the transitive closure of T.

LEMMA 5.18 *For all tournaments T: $\varepsilon(T) = d(f_{trans}(T),T)$.*

Proof Suppose that the transitive closure of T is equal to $f_{trans}(T) = X_1 \gg \ldots \gg X_k$. Now take $x,y \in X$. The pairwise comparison between x and y lies on a cycle, iff there exists an $i \in \{1,\ldots,k\}$ such that $x,y \in X_i$. Thus, if we define $t_i = |X_i|$, we have $\delta(T) = \frac{1}{N}\sum_{i=1}^{k} t_i(t_i-1)/2$, $N = n(n-1)/2$, $n = |X|$, where $t_i(t_i-1)/2$ is the number of pairwise comparisons between elements of X_i. But this expression is also equal to $\frac{1}{N}d(f_{trans}(T),T)$, hence $\varepsilon(T) = d(f_{trans}(T),T)$. ∎

Of course, in defining a circularity measure that gives the distance between a tournament and a weak order, we could have taken any ranking rule $f : \mathbf{T}(X) \to \mathbf{W}(X)$, giving $d(f(T),T)$. This is the f-induced circularity measure:

- **f-induced**

  Let f be a ranking rule on $\mathbf{T}$. A circularity γ on $\mathbf{T}$ is f-induced, if for all tournaments T: $\gamma(T) = d(f(T),T)$.

We yet introduce another f related property for a circularity measure.

- **f-additive**

  Let f be a ranking rule on $\mathbf{T}$. A circularity measure γ on $\mathbf{T}$ is f-additive, if for all tournaments T

  $\gamma(T) = \sum_{i=1}^{k} \gamma(T,X_i)$, *where* $f(T) = X_1 \gg \ldots \gg X_k$.

LEMMA 5.19 *The circularity measure* ε *on* $\mathbf{T}$ *is* f_{trans}*-additive.*

Proof Suppose that $f_{trans}(T) = X_1 \gg \ldots \gg X_k$. Since the pairwise comparison between two alternatives x and y is part of a cycle, if and only if x and y are in the same indifference class of f_{trans}, it is easy to verify that ε is f_{trans}-additive. ■

Now, given a ranking rule f, we can generate all circularity measures that are additive with respect to f: define γ for all non-empty sets $Y \subseteq U$ and all tournaments (S,Y) such that $f(S,Y) = Y$. Then let $\gamma(T)$ be recursively defined by $\sum_{i=1}^{k} \gamma(T,X_i)$, where $f(T) = X_1 \gg \ldots \gg X_k$. But, as we shall prove in theorem 5.20, under certain conditions for the ranking rule f, a circularity measure can be both f-additive and f-induced, only if $f = f_{trans}$. One of these conditions for f is that it is recursively closed:

Definition 5.9
A ranking rule f *on* $\mathbf{T}$ *is called recursively closed, if for all tournaments* T, $f^*(T) = f(T)$, *where* f^* *is the recursive application of* f.

For example, f_{trans} and f^*_{out} satisfy this condition. Now, we present theorem 5.20.

THEOREM 5.20 (Characterization of ε) *Let* f *be a ranking rule on* $\mathbf{T}$ *that is recursively closed and commutes with concatenation.*
A circularity measure γ *on* T *is* f*-induced and* f*-additive, iff* $\gamma = \varepsilon$.

Proof (if) This is proved in lemma 5.18 and 5.19.
(only if) Suppose $f(T) = X_1 \gg \ldots \gg X_k$. Let $t_i = |X_i|$, $i \in \{1,\ldots,k\}$. Because γ is f-induced, $\gamma(T) = d(f(T),T)$. This means that

$$\gamma(T) \geq \sum_{i=1}^{k} t_i(t_i-1)/2 \tag{1}$$

because all alternatives from X_i are indifferent in the ranking produced by f. It is larger than this sum if and only if there are alternatives x and y such that xTy while $y \gg x : f(T)$.

On the other hand, because γ is f-additive, we have $\gamma(T) =$

$\sum_{i=1}^{k} d(f(T,X_i),(T,X_i))$. But, since f is recursively closed, $f(T,X_i) = X_i^2$, giving $d(f(T,X_i),(T,X_i)) = t_i(t_i-1)/2$. Thus

$$\gamma(T) = \sum_{i=1}^{k} t_i(t_i-1)/2. \qquad (2)$$

Combining (1) and (2), we see that $X_i \gg X_j : T$ if $i < j$. Because f commutes with concatenation, the indifference classes of f are not a concatenation of two tournaments. Thus $f = f_{trans}$, which means $\gamma = \varepsilon$. ■

Note that we could have replaced the recursively closed condition with the condition that for all tournaments T, $d(f(T),T) \leq |X|(|X|-1)/2$. It means that the weak order $f(T)$ is not allowed to differ too much from T; the distance between them is less than or equal to the number of pairwise comparisons made.

PROPOSITION 5.21 *Let f be a ranking rule on $\mathbf{T}$ commuting with concatenation and such that for all $\varnothing \neq Y \subseteq U$ and all tournaments (T,Y): $d(f(T,Y),(T,Y)) \leq |Y|(|Y|-1)/2$.*
A circularity measure γ is f-induced and f-additive, iff $\gamma = \varepsilon$.

Proof Expression (2) in the proof of theorem 5.20 may be replaced by $\gamma(T) \leq \sum_{i=1}^{k} t_i(t_i-1)/2$. ■

Independence of the conditions in theorem 5.20.
Clearly, $rsc(T) = d(f^*_{out}(T),T)$ is f^*_{out}-induced, but not f^*_{out}-additive, as the proof of theorem 5.20 shows. If we only know that a circularity measure γ is f-additive, we are free in choosing $\gamma(S,Y)$ if $f(S,Y) = Y$.

5.6 STATISTICAL ANALYSIS

In this section, we want to answer the following questions. First, what are acceptable values for λ and i? Second, what is the rank correlation between λ and i? To answer these questions, we use statistical analysis.

5.6.1 Acceptable values

Consider tournaments on a set X with $|X| = 7$. As stated in proposition 5.1, the maximum number of 3-cycles possible is 14. Now, every value between 0 and 14 can occur. But how do we decide which values of λ are acceptable? Analogously, i ranges from 0 to 7. For what values of i does a tournament show some degree of consistency with respect to an order, not withstanding a lack of perfection? For what values does it look as if the tournament has been randomly generated?

To answer these question, we introduce H, the null hypothesis that T is randomly generated. This means that for all $\{x,y\} \subseteq X$, x and y are equally likely to be chosen from the set $\{x,y\}$. Or in other words, we are unaware of any difference between the objects. Choices are made at random, independently of one another.

This null hypothesis is rejected if an unexpectedly large number of choices are internally consistent. This means that they cohere with one out of all m! possible orders for m objects. Of course, this raises two questions. First, how do we measure internal consistency when a tournament is randomly generated? Second what is meant by 'unexpectedly large', how small must λ or i be?

In Slater (1961) a clear choice is made in favour of i measuring internal consistency or consistency to an order. Because the null hypothesis relates to all $\binom{n}{2}$ pairwise comparisons, these choices are the simple events that are to be studied. Slater rejects the use of λ, since 3-cycles are compound events, not conceivable independent of one another. For example, if n is large, $\lambda_{max} > \binom{n}{2}$. Moreover, it is clear that the use of λ implies that some inconsistencies receive more weight than others. To illustrate this, suppose we have two sets of pairwise comparisons between 5 objects $a_1,\ldots,a_5$. We assume that a_iTa_j if $i < j$, except that in the first set a_3Ta_1 and in the second set a_5Ta_1. The value of i is the same for both sets, the inconsistencies receive equal weight. Using λ, the inconsistency a_5Ta_1 receives more weight than a_3Ta_1, because a_5Ta_1 introduces more 3-cycles. But since the null hypothesis assumes that inconsistencies occur at random, there is no justification for this weighting. Only after the null hypothesis is rejected, weighting can become relevant. At first instance, all responses receive equal weight. True as this may sound, David (1988) does not entirely agree with Slater. He argues that for most reasonable alternative hypotheses H_a to randomness, the slight upset a_3Ta_1 is more likely to occur than the major upset a_5Ta_1. Therefore the first set

of pairwise comparisons, being more in accord with H_a than the second, is less compatible with H. This is reflected in its lower λ value.
Compare our characterizations of λ and i.

Hereafter, we describe a technique that we use in deciding whether or not we do reject the null hypothesis. As will become clear, this technique is applicable to both measures. Therefore, we continue to use both λ and i.

In answering the second question (what is unexpectedly large?), we assume that we have a tournament, say T, and $i(T) = i_0$.
If i_0 is small, we may have the feeling that it is unlikely that T is randomly generated. To give a more precise meaning to this feeling, we compute the probability that $i(S) \leq i_0$ for randomly generated tournaments S. This is the conditional probability $P(i \leq i_0|H)$. It is used to scale all circularity measures to the interval (0,1]:

Definition 5.10
Let γ be a circularity measure. Then γ_{scaled} is a circularity measure defined by

$$\gamma_{scaled}(T) = P(\gamma \leq \gamma(T)|H),$$

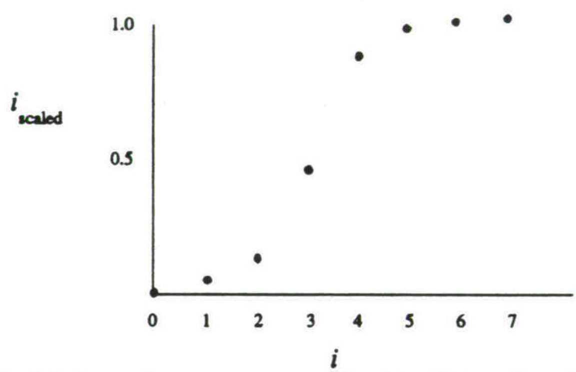
where H is the hypothesis that tournaments are randomly generated.

Compare Slater (1961), who called it the cumulative proportions.
Note that $\gamma_{scaled} : \mathbf{T}(X) \rightarrow (0,1]$.

Suppose our computation shows that $i_{scaled}(T) = p < 0.05$. Then the probability that in a random generated tournament, the value of i does not exceed $i_0 = i(T)$, is smaller than 0.05, hence is highly unlikely. Conversely, since we observed this small value $i(T) = i_0$, we may state that it is unlikely that T is randomly generated; there is some degree of consistency. In that case we reject H and replace it by an alternative hypothesis (for example by the hypothesis that we have weak stochastic transitivity, see definition 5.11).
Hence, the higher $i_{scaled}(T)$, the more likely it is for T to be randomly generated. On the other hand, if $i_0 = i(T)$ is small, $i_{scaled}(T)$ is small, and it is unlikely that T is randomly generated.
To summarize, if $i_{scaled}(T) < 0.05$, we accept the tournament and the value of i, because there is some degree of consistency. If $i_{scaled}(T)$ exceeds

0.05, it is very well possible that the tournament is randomly generated and we consequently treat it like that. Of course, this threshold value 0.05 is arbitrary and may be replaced by another value. Nevertheless, almost everyone experiences 0.05 as *the* threshold value.

To give an example, we took tournaments on X with $|X| = 7$, see graph 5.1. For precise values, we refer to table 5.2.
This figure has to read as follows. For example, if $i(T) = 1$ then $i_{scaled}(T) = 0.03$, which means that 3% of all tournaments do have a value of i smaller than or equal to 1. From this graph and table 5.2, we deduce that T is acceptable if and only if $i(T) \leq 1$, because in that case $i_{scaled}(T) \leq 0.05$ and we may reject H.



Graph 5.1 i_{scaled} for tournaments T with $|X| = 7$, c.f. Table 5.2.

To give values for λ_{scaled} and i_{scaled} when n = 4, 5, 6 and 7, consider table 5.2. The values are derived from Slater (1961). For larger values of n, we may use random samples.

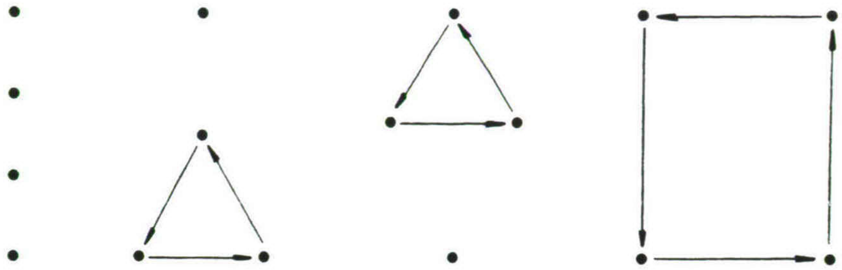
n = 4		n = 5		n = 6		n = 7			
λ	λ_{scaled}	λ	λ_{scaled}	λ	λ_{scaled}	λ	λ_{scaled}		
0	.375	0	.117	0	.022	0	.002	8	.420
1	.625	1	.234	1	.051	1	.006	9	.553
2	1	2	.469	2	.119	2	.017	10	.737
		3	.703	3	.208	3	.033	11	.853
		4	.977	4	.398	4	.069	12	.964
		5	1	5	.509	5	.112	13	.999
				6	.773	6	.198	14	1
				7	.919	7	.287		
				8	1				

n = 4		n = 5		n = 6		n = 7			
i	i_{scaled}	i	i_{scaled}	i	i_{scaled}	i	i_{scaled}		
0	.375	0	.117	0	.022	0	.002	5	.979
1	1	1	.586	1	.183	1	.030	6	.999
		2	.977	2	.588	2	.163	7	1
		3	1	3	.941	3	.474		
				4	1	4	.825		

Table 5.2 Values of λ_{scaled} and i_{scaled}, see Slater (1961).

We see that for small n, the method of setting arbitrarily a level α, for example $\alpha = 0.05$, is not very useful: if n = 5, even transitive tournaments are unacceptable: we cannot reject H. Nevertheless, the scaled values give some insight in the circularity. Moreover it makes clear another difference between i and λ. The various sets of acceptable tournament structures for a given level α, are quite different. For example, if $\alpha = 0.185$, there are tournaments T and S, such that $\lambda_{scaled}(T) < \alpha$, while $i_{scaled}(T) > \alpha$ and $\lambda_{scaled}(S) > \alpha$, while $i_{scaled}(S) < \alpha$. See table 5.3, structures 7 and 14, n = 6.

For n = 4 one may verify the values of table 5.2 by using figure 5.3, showing the four possible tournament structures and the number of ways of labeling their nodes, where it is to be understood that if an arrow has not been drawn, it is oriented from the higher node to the lower node. For example, the second structure allows for 8 different ways of labeling: for the first node we have four possible labels (a, b, c or d), and there are two different 3-cycles on the remaining three labels.



ways of labeling:

24 8 8 24

Figure 5.3 The four different tournament structures of $\tau(X)$, $|X| = 4$ and the number of ways of labeling their nodes.

All $2^{\binom{n}{2}}$ tournaments on a set X with $n = |X|$ are equally likely if the tournaments are randomly generated. Thus the probability of obtaining a tournament of structure 1, is 24/64. The other probabilities are 8/64, 8/64 and 24/64.

In determining $\lambda_{,scaled}$ for large n, we use a normal approximation: If a tournament T is randomly generated, the number of 3-cycles $\lambda(T)$ of that tournament, is approximately normally distributed if n, the number of alternatives, is sufficiently large ($n \geq 10$). Furthermore, for all n, the mean number of 3-cycles is equal to $\mu(n) = 1/4\binom{n}{3}$ and its variance is $\sigma^2(n) = 3/16\binom{n}{3}$, see Moon (1968, page 12).

As mentioned earlier, if the null hypothesis is rejected, we replace it by an alternative one, for example that we have weak stochastic transitivity, see Roberts (1979).

Definition 5.11

A system $P = (p_{ij})$ satisfies weak stochastic transitivity if for all i,j,k

$$\text{if } p_{ij} \geq 1/2 \text{ and } p_{jk} \geq 1/2 \quad \text{then} \quad p_{ik} \geq 1/2,$$

where p_{ij} is the probability that the i-th alternative beats the j-th.

Such a system gives rise to (various) so-called weak stochastic orders $(a_1,a_2,\dots,a_n)$ where $p_{ij} \geq 1/2$ if $i < j$.

The case for the nearest adjoining order has been strengthened by a probabilistic basis provided in Thompson and Remage (1964) and Remage and Thompson (1966). Let α_{ij} be 1 if alternative a_i wins from a_j, otherwise it is 0. The likelihood of a tournament T is therefore $L(T) = \prod_{i<j} p_{ij}^{\alpha_{ij}}(1-p_{ij})^{1-\alpha_{ij}}$. Thompson and Remage propose ranking the alternatives by maximizing L with respect to $P = (p_{ij})$, subject to the weak stochastic transitivity condition. They proved that the resulting weak stochastic orders are Slater's nearest adjoining orders.

On the other hand, if we maximize the likelihood with respect to the strict utility model (see definition 5.8), it is shown in chapter 3 section 3.8, that the resulting weak order is the same as $f_{out}(T)$. This may be seen as a strengthening for the case of λ.

If we have two circularity measures γ^1 and γ^2, for which there is a strictly increasing function ϕ such that $\gamma^1 = \phi\circ\gamma^2$, then $\gamma^1_{scaled} = \gamma^2_{scaled}$.
To give an example, consider Kendall's coefficient of consistency $k(T)$, which is defined as follows. Given the score vector, the mean score of the alternatives is equal to $\bar{s} = \frac{\sum_{x\in X} s_x}{n} = \frac{n-1}{2}$, where $n = |X|$. The variability of the scores in T is therefore

$$v(T) = \sum_{x\in X}(s_x - \bar{s})^2 = \sum_{x\in X} s_x^2 - \frac{n(n-1)^2}{4}.$$

Now, define $k(T) = -\, v(T)$. It is easy to verify that

$$\lambda(T) = \binom{n}{3} - n(n-1)/4 - n(n-1)^2/2 + 1/2k(T).$$

PROPOSITION 5.22 $\lambda_{scaled} = k_{scaled}$.

Proof As is described above, $\phi(k(T)) = \lambda(T)$, where

$$\phi(t) = \frac{1}{2}t + \binom{n}{3} - n(n-1)/4 - n(n-1)^2/2.$$

Because $\phi'(t) = 1/2 > 0$, $\lambda_{scaled} = k_{scaled}$. ■

5.6.2 Rank correlation

Using a circularity measure, we may order all possible tournament structures to their circularity, from low to high. For example, if n = 4, there are 4 different structures, 1 to 4, see figure 5.3. Since the corresponding values of λ are equal to: 0, 1, 1, 2, we obtain the sequence or weak order of tournament structures

$\{1\}$ $\{2,3\}$ $\{4\}$.

We may do this for i as well. The appendix in Moon (1968, page 91-95) contains all the different (or nonisomorphic) structures for n = 4, 5 and 6. If n = 4, there are 4 structures, for n = 5 we have 12 structures, finally for n = 6 there are 56 different structures. If we numerate the tournament structures that are introduced in Moon (1968) in the obvious way, we obtain the following sequences of structures, from low to high circularity, see table 5.3.

Note that strict differences between i and λ for the first time occur when n = 6. Consider the following pairs: (7,14), (8,19), (12,31), (12,32), (12,36), (12,37), (14,16), (14,19) and (17,19).

In order to compute the rank correlation for each X, we order all $2^{\binom{n}{2}}$ tournaments by using the weak orders of table 5.3 and the number of ways of labeling their nodes, compare figure 5.3. If γ is the circularity measure, this weak order is written as $s_n(\gamma)$.

For example, if n = 4, this order for λ is $\{a\} \gg \{b,c\} \gg \{d\}$, where a is an indifference class containing 24 tournaments of structure 1, b (c) is a class containing 8 tournaments of structure 2 (3) and d is an indifference class containing 24 tournaments of structure 4.

Using the weak orders $s_n(\lambda)$ and $s_n(i)$, we may compute the rank correlation between i and λ. We make use of the symmetric difference d to compute this rank correlation, see definition 5.3. The symmetric difference attains its maximal value if we take a linear order L and its reverse vL:

$d(L,vL) = n(n-1)$, $n = |X|$.

n = 4		n = 5	
i	λ	i	λ
{1}	{1}	{1}	{1}
{2,3,4}	{2,3}	{2,3,4,5,6,7}	{2,4,5}
	{4}	{8,9,10,11}	{3,6}
		{12}	{7,8}
			{9,10,11}
			{12}

n = 6	
i	λ
{1}	{1}
{2..7,13,15,16,19}	{2,3,5,13}
{8..11,14,17,18,20..32,36,37,41}	{4,6,14,15}
{12,33,34,35,38,39,40,42..52}	{7,8,16,17}
{53..56}	{9,10,11,18..21,25..27}
	{12,22..24,28..30,51}
	{31..42}
	{43..50}
	{52..56}

Table 5.3 Structures ordered by using i and λ.

Definition 5.12

Let $n = |X|$ and α, β be two circularity measures on $\mathbb{T}(X)$. Then the rank correlation $r(\alpha,\beta)$ is defined by

$$r(\alpha,\beta) = 1 - d(s_n(\alpha),s_n(\beta))/t_n(t_n-1)$$

where $t_n = 2^{\binom{n}{2}}$ is the number of different tournaments.

In case α and β are linear orderings, r reduces to Kendall's rank correlation, which is equal to the number of upsets of the one linear order with respect to the other, divided by the maximal value n(n-1). The values are given in table 5.4. For example, if n = 4, then $t_4 = 64$, d = 2x8x24, thus $r(\lambda,i) = 0.905$. We also give the rank correlations for the circularity measures, λ, i, ρ and rsc, where $rsc(T) = d(f^*_{out}(T),T)$.

Although the rank correlation is pretty high, it does not prevent differences between λ and i as described after table 5.2.

See Bezembinder (1981) for the comparison between λ, i and ρ by looking at their product moment correlation

$$\mathrm{corr}(\alpha,\beta) = E((\alpha-\mu_\alpha)(\beta-\mu_\beta))/\sigma_\alpha\sigma_\beta.$$

If $\mathrm{corr}(\alpha,\beta)$ is close to 1, it means that if α is large (with respect to its mean) then β is large, and vice versa.

n = 4

	λ	i	ρ	*rsc*
λ	1	.905	1	.666
i		1	.905	.833
ρ			1	.666

n = 5

	λ	i	ρ	*rsc*
λ	1	.885	.924	.811
i		1	.874	.894
ρ			1	.742

n = 6

	λ	i	ρ	*rsc*
λ	1	.865	.766	.813
i		1	.765	.835
ρ			1	.714

Table 5.4 The rank correlations for n = 4, 5 and 6.

5.7 TABLE OF PROPERTIES

In this section, we summarize some properties of circularity measures considered in this chapter.

We start this section with the introduction of just one additional property. We first repeat the definition of a separable tournament, compare chapter 2.

Definition 5.13 (Storcken (1989))
Suppose $T \in \mathbf{T}(X)$. Then a tournament (T,B), $B \subseteq X$ is called separable from (T,X) in T if for all $x \in X\backslash B$: bTx for all $b \in B$ or xTb for all $b \in B$.
This means that the elements of B behave like a single alternative.

We consider a circularity measure γ on $\mathbf{T}$ and a tournament $T \in \mathbf{T}(X)$. For all separable subsets B of X, we consider the set of tournaments T' such that $T' \cap (X^2-B^2) = T \cap (X^2-B^2)$. Thus changes are limited to B^2. Furthermore, we demand that the replacement of (T,B) with (T',B) is such that $\gamma(T,B) = \gamma(T',B)$:

- **Separability**

  A circularity measure γ on $\mathbf{T}$ is separable, if for all T and all separable subsets B of X we have:

  if $T' \cap (X^2-B^2) = T \cap (X^2-B^2)$ and $\gamma(T',B) = \gamma(T,B)$ then $\gamma(T') = \gamma(T)$.

PROPOSITION 5.23 *The measures λ, i, ε, δ, ρ and χ are separable.*

Proof (λ) Suppose B is separable from X in T. Take a 3-cycle. If it is entirely contained in B there is no problem, because $T \cap B^2$ is replaced by a tournament $T' \cap B^2$ with the same number of 3-cycles. If it is entirely contained in X\B, we also have no problem. Moreover, it is impossible that 2 of the 3 elements are in B, because these two elements behave uniform with respect to elements outside B. Hence only one of the three alternatives is in B, which means that this 3-cycle can not be removed.
(i) Suppose that (T,B) is separable from T. Using an indirect proof, it is clear that, if L is a nearest adjoining order of T, then (L,B) is so for (T,B). From this it follows that i is separable.
(ε, δ, ρ, χ) Left to the reader. ■

In the following table, the ranking rule f is recursively closed and commutes with concatenation.

	standard-ization	reduc-ible	separ-able	exten-sive	equable	outscore equable	indepen-dent of 3-cycles	f-induced &-additive
λ	■	■	■	■	□	◉	■	□
i	■	■	■	■	◉	□	□	□
ε	■	□	■	■	□	□	■	◉
δ	■	□	■	□	□	□	■	□
ρ	■	□	■	■	□	□	■	□
χ	■		■	■	□	□	□	□

□: not satisfied

■: satisfied

◉ : characterizing properties, to be taken together in each row.

Table 5.5 Table of properties.

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SUBJECT INDEX

KEUZE, ORDENING EN CIRCULARITEIT IN ASYMMETRISCHE RELATIES

SAMENVATTING

In dit proefschrift bestuderen we verzamelingen van paarsgewijze vergelijkingen. Hierbij worden alternatieven geëvalueerd in termen van bijvoorbeeld preferentie, belangrijkheid, dominantie, kwaliteit enz. We nemen aan dat als er in de vergelijking van een tweetal alternatieven een strikte keuze voor een van beide wordt gemaakt, deze keuze definitief is. Dit gebeurt bijvoorbeeld wanneer alternatieven en criteria voldoende zijn uitgekristalliseerd, zodat de gemaakte keuze kan worden gemotiveerd. Gebruikmakend van deze paarsgewijze vergelijkingen, trachten we de alternatieven te ordenen. In het algemeen echter, wordt dit bemoeilijkt door het optreden van intransitiviteiten in het kiezen tussen alternatieven. Desondanks blijft in veel gevallen de noodzaak tot het verkrijgen van een ordening bestaan. Vandaar dat ons eerste hoofdthema is: hoe moeten we alternatieven ordenen, hoe kiezen we een beste alternatief, hoe bepalen we een dominante coalitie? Daartoe worden een groot aantal ordenings methoden geintroduceerd en besproken. Nu is het zo dat het moeilijk is om te beslissen welke methode moet worden gebruikt, want elke methode heeft wel enkele aantrekkelijke eigenschappen. Natuurlijk kunnen we trachten het toepassingsgebied van elk der methoden te bepalen. Een methode die bijvoorbeeld geschikt is om spelers in een toernooi te rangschikken, zal wellicht ongeschikt blijken voor problemen met een politiek karakter. Onze benadering van het probleem is derhalve axiomatisch van aard. We bestuderen een groot aantal condities die kunnen worden gebruikt bij het kwalitatief vergelijken van ordeningsmethoden. Deze condities representeren, in mindere of meerdere mate, de noties van redelijkheid en eerlijkheid. Gebruikmakend van deze condities kan in sommige gevallen een karakterisering worden afgeleid. Dit is een collectie van onderling onafhankelijke condities voor een klasse van methoden, zodat er slechts één methode overblijft die aan alle condities voldoet. Op die manier kunnen we in een aantal gevallen aangeven wanneer een bepaalde methode kan worden toegepast.

Het tweede hoofdthema is de bepaling van de inconsistentie of circulariteit van een toernooi. Een circulariteitsmaat geeft aan in hoeverre een toernooi

kan worden geordend of wat de afstand is tot een ordening. Net als in het geval van ordeningsregels en keuzefuncties, bestaat ook hier geen maat die als beste wordt erkend. Onze benadering van het probleem is axiomatisch van aard. We karakteriseren circulariteitsmaten met behulp van noodzakelijke en voldoende condities.

De indeling van het proefschrift is als volgt.

Hoofdstuk 1 valt uiteen in drie gedeelten. In het eerste gedeelte worden een aantal illustratieve voorbeelden besproken. Het tweede gedeelte bevat enige elementaire stellingen betreffende de relatie tussen irreducibiliteit en cycliciteit. Tevens worden noodzakelijke begrippen ingevoerd, zoals binaire relaties, toernooien, ordeningen, transitieve afsluiting, permutaties enz. De appendix geeft een overzicht van de ordenings en keuze procedures, circulariteitsmaten en condities die zijn gebruikt in een kwalitatieve studie.

In hoofdstuk 2 behandelen we keuzefuncties voor toernooien. Als eerste keuzefunctie bespreken we de topcykel. We laten zien dat deze functie een aantal gebreken heeft. Ten eerste is de resulterende keuzeverzameling vaak zo groot dat deze moet worden verfijnd, verder kan de topcykel Pareto gedomineerde elementen bevatten. Beide bezwaren zijn aanleiding om andere keuzemechanismen te introduceren. We bespreken de uncovered set, de minimal covering set en karakteriseringen hiervoor. Daarnaast bespreken we de Banks set, de tournament equilibrium set, de Copeland keuzefunctie en de Slater keuzefunctie. We geven een nieuw bewijs van de uniciteit van de optimale strategie van een toernooi-nulsomspel. Als toepassing hiervan, leiden we een karakterisering af van de corresponderende keuzefunctie. We voeren een groot aantal nieuwe condities in zoals Δ-IIA, separabiliteit, globale monotoniciteit, staartstabiliteit enz. Het hoofdstuk wordt afgesloten met een tweetal tabellen betreffende een aantal verzameling-theoretische inclusies, en een aantal eigenschappen van keuzefuncties.

Ordeningsregels worden uitgebreid onderzocht in hoofdstuk 3. Ook hier introduceren we een groot aantal nieuwe condities die kunnen worden gebruikt bij het kwalitatief vergelijken van verschillende ordeningsregels. We geven een karakterisering van door keuzefuncties geinduceerde ordeningsregels, de transitieve afsluiting en de Copeland ordeningsregel. Daarnaast karakteriseren we de Copeland keuzefunctie en een daaraan gerelateerde keuzefunctie. Ook regels die zijn gebaseerd op een afstandsbegrip worden bestudeerd. In een aparte paragraaf worden door

ratings geinduceerde ordeningen onderzocht. Tevens worden twee condities besproken die het verschil tussen twee nivo's van analyse beschrijven. Een tabel aan het einde van dit hoofdstuk bevat een samenvatting van een systematisch onderzoek van ordeningsregels en eigenschappen. In de eerste appendix wordt een deelverzameling van de verzameling van toernooien bepaald waarop de Copeland ordeningsregel voldoet aan de Δ-IIA conditie. De tweede appendix bespreekt het effect van het toevoegen van nieuwe alternatieven op de de inconsistentiemaat die wordt gebruikt door Saaty in zijn 'Analytic Hierarchy Process'.
In hoofdstuk 4 nemen we aan dat de asymmetrische relatie een dominantiestructuur representeert. Een bekend oplossingsmechanisme hierbij is de zogenaamde von Neumann-Morgenstern stabiele verzameling (vNM-stabiele verzameling). Het combineert de noties van interne en externe stabiliteit van een coalitie. Het eerste hoofdresultaat is een karakterisering van de keuzefunctie die correspondeert met deze vNM-stabiele verzameling in het geval van acyclische relaties. Sommige cyclische relaties hebben geen vNM-stabiele verzameling. Daarom introduceren we vervolgens gegeneraliseerde vNM-stabiele verzamelingen. Het tweede hoofdresultaat van dit hoofdstuk krijgen we door gebruikmaking van spel-theoretische argumenten. Dit betreft een karakterisering van een unieke gegeneraliseerde vNM-stabiele verzameling die voldoet aan de eigenschap van de zogenaamde gebalanceerde dominantie. Tenslotte leiden we enige verzameling-theoretische inclusies af voor een aantal keuzefuncties.
In het laatste hoofdstuk gaan we in op de circulariteit van een toernooi. Het zwaartepunt van de bespreking hiervan ligt bij een vergelijkende studie van twee circulariteitsmaten: het aantal 3-cykels λ en Slaters i. Gebruikmakend van een onmogelijkheidsstelling laten we zien dat er geen maat is die alle aantrekkelijke eigenschappen van zowel λ als i in zich verenigt. Vandaar dat voor beide maten afzonderlijk een karakterisering wordt gegeven. Hiermee kan een indicatie van het toepassingsbebied van beide circulariteitsmaten worden gegeven. Daarnaast karakteriseren we circulariteitsmaten die zijn gedefinieerd in termen van scores van de alternatieven en laten we zien dat dit correspondeert met het zogenaamde 'strict utility model'. Verder karakteriseren we een circulariteitsmaat die wordt geinduceerd door de transitieve afsluiting. Tenslotte geven we een statistische analyse en besluiten we het hoofdstuk met een tabel van eigenschappen voor de verschillende maten.

CURRICULUM VITAE

Herman Monsuur werd geboren op 7 oktober 1961 in Genemuiden, waar hij tevens zijn jeugd doorbracht. In 1980 behaalde hij het VWO diploma aan het Johannes Calvijn Lyceum te Kampen. In datzelfde jaar begon hij met een studie wiskunde aan de Rijksuniversiteit Groningen. Na het kandidaatsexamen in 1984 behaalde hij in 1987 het doctoraalexamen met als specialisatie Dynamische systemen. De afstudeerscriptie werd geschreven onder begeleiding van Prof. Takens. Vanaf 1988 is de auteur van dit proefschrift werkzaam als universitair docent aan het Koninklijk Instituut voor de Marine te Den Helder. Tot 1992 in de vakgroep Wis- en natuurkunde, vanaf 1993 in de vakgroep Internationale Veiligheidsstudies. In 1989 startte hij zijn onderzoek op het gebied van paarsgewijze vergelijkingen en ordenings procedures. De resultaten van dit onderzoek, verricht onder begeleiding van Prof. Delver, Prof. Tijs en Dr. Storcken, zijn terug te vinden in dit proefschrift.